Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Conditions for the Complete Reducibility of Groups of Linear Substitutions.

By Henry Taber.

§ 1. *Conditions Sufficient for the Complete Reducibility of the Group G.*

Let $A_1, A_2, \dots, A_m$ be any system of linearly independent linear homogeneous transformations in n variables ($m \leqq n^2$) constituting a hypercomplex number system so that

$$A_i A_j = \Sigma_{k=1}^{m} \gamma_{ijk} A_k, \qquad (i, j = 1, 2, \dots, m). \tag{1}$$

Let A_i, for i equal successively to $1, 2, \dots, m$, be defined by the system of equations

$$\xi'_\mu = \Sigma_{\nu=1}^{n} \alpha_{\mu\nu}^{(i)} \xi_\nu, \qquad (\mu = 1, 2, \dots, n), \tag{2}$$

or

$$(\xi'_1, \xi'_2, \dots, \xi'_n) = \left|\begin{matrix} \alpha_{11}^{(i)}, & \alpha_{12}^{(i)}, & \dots, & \alpha_{1n}^{(i)} \\ \alpha_{21}^{(i)}, & \alpha_{22}^{(i)}, & \dots, & \alpha_{2n}^{(i)} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1}^{(i)}, & \alpha_{n2}^{(i)}, & \dots, & \alpha_{nn}^{(i)} \end{matrix}\right| (\xi_1, \xi_2, \dots, \xi_n) \tag{3}$$

Then, for any two integers i, j from 1 to m,

$$\Sigma_{\lambda=1}^{n} \alpha_{\mu\lambda}^{(i)} \alpha_{\lambda\nu}^{(j)} = \Sigma_{k=1}^{m} \gamma_{ijk} \alpha_{\mu\nu}^{(k)}, \qquad (\mu, \nu = 1, 2, \dots, n). \tag{4}$$

By equation (1),

$$\begin{aligned} \Sigma_{i=1}^{m} x_i A_i \cdot \Sigma_{j=1}^{m} y_j A_j &= \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} x_i y_j A_i A_j \\ &= \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} (x_i y_j \Sigma_{k=1}^{m} \gamma_{ijk} A_k) \\ &= \Sigma_{k=1}^{m} (\Sigma_{i=1}^{m} \Sigma_{j=1}^{m} x_i y_j \gamma_{ijk}) A_k. \end{aligned} \tag{5}$$

Whence it follows that the aggregate of numbers of the hypercomplex number system $(A_1, A_2, \dots, A_m)$, that is, the aggregate of transformations

$$X = x_1 A_1 + x_2 A_2 + \dots + x_m A_m, \tag{6}$$

for every system of scalars $x_1, x_2, \dots, x_m$, constitutes a group G with m essential parameters.[*] Therefore, *any linear function of transformations of G*

---

[*] For, if $\Sigma_{i=1}^{m} x_i A_i = \Sigma_{i=1}^{m} y_i A_i$, then $\Sigma_{i=1}^{m} (x_i - y_i) A_i = 0$; and, therefore, $x_i - y_i = 0$ for $i = 1, 2, \dots, m$, since, otherwise, the A's are not linearly independent, which is contrary to supposition.

is also a transformation of this group. The general transformation $X=\Sigma_{i=1}^m x_i A_i$ of G is defined by the system of equations

$$\xi'_\mu=\Sigma_{i=1}^m \Sigma_{\nu=1}^n x_i \alpha_{\mu\nu}^{(i)} \xi_\nu, \qquad (\mu=1, 2, \ldots, n), \tag{7}$$

or

$$(\xi'_1, \xi'_2, \ldots, \xi'_n) = \begin{array}{l} (\Sigma_{i=1}^m x_i \alpha_{11}^{(i)}, \Sigma_{i=1}^m x_i \alpha_{12}^{(i)}, \ldots, \Sigma_{i=1}^m x_i \alpha_{1n}^{(i)} \big)\!\big(\xi_1, \xi_2, \ldots, \xi_n) \\ \left| \begin{array}{l} \Sigma_{i=1}^m x_i \alpha_{21}^{(i)}, \Sigma_{i=1}^m x_i \alpha_{22}^{(i)}, \ldots, \Sigma_{i=1}^m x_i \alpha_{2n}^{(i)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Sigma_{i=1}^m x_i \alpha_{n1}^{(i)}, \Sigma_{i=1}^m x_i \alpha_{n2}^{(i)}, \ldots, \Sigma_{i=1}^m x_i \alpha_{nn}^{(i)} \end{array} \right| \end{array} \tag{8}$$

I shall denote the determinant of X by $|X|$; and I shall assume that $|X| \neq 0$ for every transformation of G. Whence it follows that the group G, that is, the aggregate of the numbers of the system $(A_1, A_2, \ldots, A_m)$ contains the identical transformation, which will be denoted simply by 1;* and, if $|X| \neq 0$, that G contains X^{-1}. For, if

$$A=a_1 A_1+a_2 A_2+\ldots+a_m A_m \tag{9}$$

is any transformation of G, we have, by Cayley's theorem,

$$\phi(A) \equiv A^n-p_1 A^{n-1}+\ldots \mp p_{n-1} A \pm p_n=0, \tag{10}$$

where

$$\phi(\rho) \equiv |\rho-A| \equiv \left| \begin{array}{llll} \rho-\Sigma_{i=1}^m a_i \alpha_{11}^{(i)}, & -\Sigma_{i=1}^m a_i \alpha_{12}^{(i)}, & \ldots, & -\Sigma_{i=1}^m a_i \alpha_{1n}^{(i)} \\ -\Sigma_{i=1}^m a_i \alpha_{21}^{(i)}, & \rho-\Sigma_{i=1}^m a_i \alpha_{22}^{(i)}, & \ldots, & -\Sigma_{i=1}^m a_i \alpha_{2n}^{(i)} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ -\Sigma_{i=1}^m a_i \alpha_{n1}^{(i)}, & -\Sigma_{i=1}^m a_i \alpha_{n2}^{(i)}, & \ldots, & \rho-\Sigma_{i=1}^m a_i \alpha_{nn}^{(i)} \end{array} \right|, \dagger \tag{11}$$

and thus $p_n=|A|$. Therefore, if $|A| \neq 0$,

$$1=\frac{\mp 1}{p_n}(A^n-p_1 A^{n-1}+\ldots \mp p_{n-1} A),$$

$$A^{-1}=\frac{\mp 1}{p_n}(A^{n-1}-p_1 A^{n-2}+\ldots \mp p_{n-1});$$

and thus 1 and A^{-1}, being linear in powers of A, are transformations of G.

Let $\rho_1, \rho_2, \ldots, \rho_r$ respectively of multiplicity $\eta_1, \eta_2, \ldots, \eta_r$, be the distinct roots of the characteristic equation, $\phi(\rho) \equiv |\rho-A|=0$, of $A=\Sigma_{i=1}^m a_i A_i$; in which case,

$$\phi(\rho) \equiv (\rho-\rho_1)^{\eta_1} (\rho-\rho_2)^{\eta_2} \ldots (\rho-\rho_r)^{\eta_r}. \tag{12}$$

I shall denote by $\bar{r}$ the greatest number of distinct roots of $\phi(\rho)=0$ for any transformation A of G. Since G contains the identical transformation, there is at least one transformation of G whose characteristic equation has $\bar{r}$ distinct

---

* The transformation $\rho 1$, defined by the system of equations

$$\xi'_\mu=\rho\xi_\mu \qquad (\mu=1, 2, \ldots, n),$$

will, following Cayley, be denoted simply by ρ, and will be identified with the scalar ρ.

† *Phil. Trans.*, Vol. CXLVIII (1858), p. 24.

non-zero roots. For, let A be so chosen that $r=\bar{r}$. Since 1 is a transformation of G, so also is $A+\lambda$ for any scalar λ. But the roots of the characteristic equation of $A+\lambda$ are given by $\rho_h+\lambda$ for $h=1, 2, \ldots, \bar{r}$; and these $\bar{r}$ roots will be distinct and other than zero provided λ is properly chosen. Wherefore, *$\bar{r}$ is equal to the greatest number of distinct non-zero roots of $\phi(\rho)=0$ for any transformation A of G.*

We may transform the hypercomplex number system $(A_1, A_2, \ldots, A_m)$ by the substitution of new units, and express the general transformation of G in terms of the latter. Thus, if

$$A'_i=\tau_{i1}A_1+\tau_{i2}A_2+\ldots+\tau_{im}A_m, \qquad (i=1, 2, \ldots, m), \tag{13}$$

the determinant of transformation not being zero, then

$$\begin{aligned} X&=x_1A_1+x_2A_2+\ldots+x_mA_m \\ &=x'_1A'_1+x'_2A'_2+\ldots+x'_mA'_m, \end{aligned} \tag{14}$$

the x''s being determined by the system of equations

$$x'_1\tau_{1j}+x'_2\tau_{2j}+\ldots+x'_m\tau_{mj}=x_j, \qquad (j=1, 2, \ldots, m). \tag{15}$$

We then have

$$A'_iA'_j=\Sigma_{i=1}^{m}\gamma'_{ijk}A'_k, \qquad (i, j=1, 2, \ldots, m), \tag{16}$$

the γ''s being linear functions of the γ's. Any linear function of the A''s is a transformation of G.

Let $A=\Sigma_{i=1}^{m}a_iA_i$ be any given transformation of G, or number of the hypercomplex number system $(A_1, A_2, \ldots, A_m)$. I shall employ S_1A, S_2A, in designation the *first* and *second scalar* of A respectively, to denote those functions of the coefficients $a_1, a_2, \ldots, a_m$ and the constants γ_{ijk} of multiplication of the system defined as follows:

$$S_1A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{ijj}, \quad S_2A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{jij}. \tag{17}$$

These functions constitute generalizations of the scalar function of quaternions, to which they reduce, becoming identical, when $m=4$ and when the system (A_1, A_2, A_3, A_4) is equivalent to that constituted by the four units of quaternions. They conform to the following theorem:

THEOREM I. *Let $A_1, A_2, \ldots, A_m$ be any system of linearly independent linear homogeneous transformations in n variables constituting a hypercomplex number system, so that*

$$A_iA_j=\Sigma_{k=1}^{m}\gamma_{ijk}A_k, \qquad (i, j=1, 2, \ldots, m);$$

and let G denote the group constituted by the aggregate of transformations

$$A=a_1A_1+a_2A_2+\ldots+a_mA_m$$

for every system of scalars $a_1, a_2, \ldots, a_m$. *Let*

$$S_1A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{ijj}, \quad S_2A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{jij}.$$

Then both S_1A *and* S_2A *are invariant to any linear transformation of the number system* $(A_1, A_2, \ldots, A_m)$; *that is to say, if*

$$A_i'=\tau_{i1}A_1+\tau_{i2}A_2+\ldots+\tau_{im}A_m, \qquad (i=1, 2, \ldots, m),$$

(*the determinant of the transformation not being zero*),

$$A_i'A_j'=\Sigma_{k=1}^{m}\gamma_{ijk}'A_k', \qquad (i, j=1, 2, \ldots, m),$$

and if

$$A=a_1A_1+a_2A_2+\ldots+a_mA_m=a_1'A_1'+a_2'A_2+\ldots+a_m'A_m,$$

then

$$S_1A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{ijj}=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i'\gamma_{ijj}',$$

$$S_2A=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i\gamma_{jij}=\frac{1}{m}\Sigma_{i=1}^{m}\Sigma_{j=1}^{m}a_i'\gamma_{jij}'.$$

If ρ *is any scalar, and*

$$B=b_1A_1+b_2A_2+\ldots+b_mA_m$$

is any second transformation of G, *then*

$$\begin{aligned} S_1\rho A &=\rho S_1A, & S_2\rho A &=\rho S_2A, \\ S_1(A\pm B) &=S_1A\pm S_1B, & S_2(A\pm B) &=S_2A\pm S_2B, \\ S_1(AB) &=S_1(BA), & S_2(AB) &=S_2(BA); \end{aligned}$$

and, if ε *is a modulus of the system,*

$$S_1\varepsilon=1, \quad S_2\varepsilon=1.$$

Let $\bar{r}$ *denote the maximum number of distinct non-zero roots possessed by the characteristic equation of any transformation of* G. *If, for any positive integer* p, *either*

$$S_1A^{p+h}=0, \qquad (h=0, 1, 2, \ldots, \bar{r}-1),$$

or

$$S_2A^{p+h}=0, \qquad (h=0, 1, 2, \ldots, \bar{r}-1),$$

A *is nilpotent; conversely, if* A *is nilpotent,*

$$S_1A^p=0, \quad S_2A^p=0,$$

for every positive integer p. *If* A *is idempotent, there is a system of just* $mS_1A>0$ *linearly independent transformations of* G *satisfying the equation*

$$AX=X,$$

in terms of which every transformation of G *satisfying this equation can be* *expressed linearly, and a system of just $mS_2A>0$ linearly independent transformations of G satisfying the equation*

$$XA=X,$$

*in terms of which every transformation of G satisfying this equation can be expressed linearly.**

I shall denote by $T_{\mu\nu}$, for $\mu, \nu=1, 2, \ldots, n$, the transformation

$$\xi_1'=0, \ldots, \xi_{\mu-1}'=0, \quad \xi_\mu'=\xi_\nu, \quad \xi_{\mu+1}'=0, \ldots, \xi_n'=0, \tag{18}$$

the coefficients of whose matrix are all zero except that in the μ-th row and ν-th column, which is equal to unity. The n^2 transformation which we obtain by giving μ and ν all integer values from 1 to n are linearly independent; and in terms of these n^2 transformations any linear homogeneous transformation in n variables can be expressed linearly. In particular, we have

$$1=T_{11}+T_{22}+\ldots\ldots+T_{nn}, \tag{19}$$

$$A_i=\Sigma_{\mu=1}^n \Sigma_{\nu=1}^n \alpha_{\mu\nu}^{(i)} T_{\mu\nu}, \qquad (i=1, 2, \ldots, m), \tag{20}$$

$$A=\Sigma_{i=1}^m a_i A_i=\Sigma_{i=1}^m \Sigma_{\mu=1}^n \Sigma_{\nu=1}^n a_i \alpha_{\mu\nu}^{(i)} T_{\mu\nu}. \tag{21}$$

Moreover,

$$T_{\mu\lambda} T_{\lambda\nu}=T_{\mu\nu}, \quad T_{\mu\lambda} T_{\lambda'\nu}=0, \qquad (\mu, \lambda, \lambda', \nu=1, 2, \ldots, n;\ \lambda' \neq \lambda); \tag{22}$$

and thus these n^2 transformations constitute a *quadrate*, that is, one of a special class of hypercomplex number systems. The first and second scalar functions for a number system constituting a quadrate are equal for any given number of the quadrate; and, therefore, but a single symbol is requisite for the scalar functions in the case of a quadrate. I shall write $\bar{S}T$ to denote the scalar of any number

$$T=\Sigma_{\mu=1}^n \Sigma_{\nu=1}^n c_{\mu\nu} T_{\mu\nu} \tag{23}$$

---

* See papers by the author in the *Trans. Am. Math. Soc.*, Vol. V (1904), p. 522, and *Proc. Am. Acad. Arts and Scs.*, Vol. XLI (1905), p. 61, Vol. XLVIII (1913), p. 628. In the first two of these papers the propositions in the above theorem are established, except that it is not there shown that $A=\Sigma_{i=1}^m a_i A_i$ is nilpotent if $S_1A^{p+h}=0$ or $S_2A^{p+h}=0$ for $h=0, 1, \ldots, \bar{r}-1$. This may be proved as follows: Let $\rho_1, \rho_2, \ldots, \rho_s$, respectively of multiplicity $\eta_1, \eta_2, \ldots, \eta_s$, be the distinct non-zero roots of $\phi(\rho) \equiv |\rho-A|=0$, the characteristic equation of the matrix A; and let

$$\psi(\rho) \equiv \rho^\nu - q_1\rho^{\nu-1} + \ldots \pm q_\nu = 0$$

be the reduced characteristic equation of A. Then

$$\psi(\rho) \equiv \rho^{\kappa_0}(\rho-\rho_1)^{\kappa_1} \ldots (\rho-\rho_s)^{\kappa_s},$$

where $1 \leqq \kappa_i \leqq \eta_i\ (i=1, 2, \ldots, s)$ and $0 \leqq \kappa_0 \leqq n-\Sigma_{i=1}^s \eta_i$. The syzygy of lowest order between powers of A, including $A^\circ=1$, is $\psi(A)=0$. Therefore, the syzygy of lowest order between powers of A, excluding the zero-th power, or 1, is $\Omega(A) \equiv \psi(A)=0$ if $\kappa_0 \neq 0$, otherwise, is $\Omega(A) \equiv A\psi(A)=0$. If now $\bar{r}$ denotes the maximum number of distinct non-zero roots of $\phi(\rho)=0$ for any number A of the number system $(A_1, A_2, \ldots, A_m)$, it is then the maximum number of distinct non-zero roots of the equation $\psi(\rho)=0$, and, therefore, the maximum number of distinct non-zero roots of $\Omega(\rho)=0$ for any number A of the system. But in the third of the papers cited above, pp. 635–637, it is shown that, if $\bar{r}$ is the maximum number of distinct non-zero roots of $\Omega(\rho)=0$ for any number A of the system, then A is nilpotent if $S_1A^{p+h}=0$ or $S_2A^{p+h}=0$ for $h=0, 1, \ldots, \bar{r}-1$.

A number A of a hypercomplex number system is *nilpotent* if $A \neq 0$, but $A^p=0$ for some positive number p; A is *idempotent* if $A^2=A \neq 0$.

of the quadrate. By (22), it follows from *Theorem I* that

$$\bar{S}T_{\mu\mu}=\frac{1}{n},\quad \bar{S}T_{\mu\nu}=0,\qquad (\mu,\nu=1,2,\ldots,n;\ \nu\neq\mu).^{*} \tag{24}$$

Therefore,

$$\bar{S}T=\Sigma_{\mu=1}^{n}\Sigma_{\nu=1}^{n}c_{\mu\nu}\bar{S}T_{\mu\nu}=\frac{1}{n}\Sigma_{\mu=1}^{n}c_{\mu\mu}. \tag{25}$$

Since $n\bar{S}T$ is the sum of the constituents in the principal diagonal of the matrix of T, it follows that $n\bar{S}T$ is equal to the sum of the roots of the characteristic equation, $|\rho-T|=0$ of T.† If T is idempotent, the roots of its characteristic equation are 0 and 1. Wherefore, *if T is idempotent, $n\bar{S}T$ is equal to the multiplicity of the root* 1 *of its characteristic equation.*

The symbol $\bar{S}$ is significant when applied to any transformation of G, as this group is a subgroup of the general linear homogeneous group; and we have

$$\bar{S}A_i=\frac{1}{n}\Sigma_{\mu=1}^{n}\alpha_{\mu\mu}^{(i)},\qquad (i=1,2,\ldots,m), \tag{26}$$

$$\bar{S}A=\frac{1}{n}\Sigma_{i=1}^{m}\Sigma_{\mu=1}^{n}a_i\alpha_{\mu\mu}^{(i)}. \tag{27}$$

Let now

$$\left.\begin{aligned}\Delta_1&\equiv\begin{vmatrix}S_1A_1A_1, & S_1A_1A_2, & \ldots, & S_1A_1A_m\\ S_1A_2A_1, & S_1A_2A_2, & \ldots, & S_1A_2A_m\\ \ldots & \ldots & \ldots & \ldots\\ S_1A_mA_1, & S_1A_mA_2, & \ldots, & S_1A_mA_m\end{vmatrix},\\ \Delta_2&\equiv\begin{vmatrix}S_2A_1A_1, & S_2A_1A_2, & \ldots, & S_2A_1A_m\\ S_2A_2A_1, & S_2A_2A_2, & \ldots, & S_2A_2A_m\\ \ldots & \ldots & \ldots & \ldots\\ S_2A_mA_1, & S_2A_mA_2, & \ldots, & S_2A_mA_m\end{vmatrix},\end{aligned}\right\} \tag{28}$$

---

[*] Let $\bar{S}_1T$ and $\bar{S}_2T$ denote the first and second scalar with respect to the quadrate of any number T of the quadrate. For any two distinct integers μ, ν from 1 to n, $T_{\mu\nu}$ is nilpotent by (22); and, therefore,

$$\bar{S}_1T_{\mu\nu}=0=\bar{S}_2T_{\mu\nu},\qquad (\mu\neq\nu)$$

by *Theorem I.* Moreover, for $1\leqq\mu\leqq n$, $T_{\mu\mu}$ is idempotent; and there is a system of just n linearly independent numbers of the quadrate, namely, $T_{\mu1}$, $T_{\mu2}$, ..., $T_{\mu n}$, satisfying the equation $T_{\mu\mu}T=T$, and in terms of which every number satisfying this equation can be expressed linearly. Similarly, there are just n linearly independent numbers of the quadrate satisfying the equation $TT_{\mu\mu}=T$, and in terms of which every number satisfying this equation can be expressed linearly. Therefore, by *Theorem I,*

$$n^2\bar{S}_1T_{\mu\mu}=n=n^2\bar{S}_2T_{\mu\mu}.$$

Wherefore, if

$$T=\Sigma_{\mu=1}^{n}\Sigma_{\nu=1}^{n}c_{\mu\nu}T_{\mu\nu},$$

then

$$\bar{S}_1T=\frac{1}{n}\Sigma_{\mu=1}^{n}c_{\mu\mu}=\bar{S}_2T.$$

† See p. 338.

denote, respectively, the resultants of the systems of equations

$$S_1XA_i = x_1S_1A_1A_i + x_2S_1A_2A_i + \ldots\ldots + x_mS_1A_mA_i = 0, \quad (i=1, 2, \ldots\ldots, m), \tag{29}$$

and

$$S_2XA_i = x_1S_2A_1A_i + x_2S_2A_2A_i + \ldots\ldots + x_mS_2A_mA_i = 0, \quad (i=1, 2, \ldots\ldots, m); \tag{30}$$

and let

$$\nabla \equiv \begin{vmatrix} \bar{S}A_1A_1, & \bar{S}A_1A_2, & \ldots\ldots, & \bar{S}A_1A_m \\ \bar{S}A_2A_1, & \bar{S}A_2A_2, & \ldots\ldots, & \bar{S}A_2A_m \\ \ldots & \ldots & \ldots & \ldots \\ \bar{S}A_mA_1, & \bar{S}A_mA_2, & \ldots\ldots, & \bar{S}A_mA_m \end{vmatrix} \tag{31}$$

denote the resultant of the system of equations

$$\bar{S}XA_i = x_1\bar{S}A_1A_i + x_2\bar{S}A_2A_i + \ldots\ldots + x_m\bar{S}A_mA_i = 0, \quad (i=1, 2, \ldots\ldots, m). \tag{32}$$

In the *Proceedings of the American Academy of Arts and Sciences*, Vol. XLVIII (1913), p. 627 *et seq.*, I have shown that $\Delta_1 = \Delta_2$, and that any number $X = \Sigma_{i=1}^m x_iA_i$ of the system $(A_1, A_2, \ldots\ldots, A_m)$, that is, any transformation X of G satisfying either one of the three systems of equations (29), (30), or (32), is also a solution of the other two; whence it follows that the nullities of Δ_1, Δ_2 and ∇ are equal. I have also shown that the aggregate of numbers of the number system $(A_1, A_2, \ldots\ldots, A_m)$ satisfying either one of these three systems of equations constitute a maximum invariant nilpotent subsystem of $(A_1, A_2, \ldots\ldots, A_m)$; that the condition necessary and sufficient that the number system shall contain no invariant nilpotent subsystem, in which case the system is either *simple* or *semi-simple* (Cartan), is $\Delta_1 = \Delta_2 \neq 0$ or $\nabla \neq 0$; and that, when this condition is satisfied, $S_1A = S_2A$ for any transformation A of G.*

I shall now assume that $\Delta_1 = \Delta_2 \neq 0$, in which case the number system $(A_1, A_2, \ldots\ldots, A_m)$ containing no invariant nilpotent subsystem, is constituted

---

* See *Theorems II, IV,* and *V* of the paper cited above, also pp. 650, 655–656. It is also there shown that the equations $\Delta_1 = 0$, $\Delta_2 = 0$, $\nabla = 0$ are invariant to any transformation of the units of the number system $(A_1, A_2, \ldots\ldots, A_m)$. Thus, if this number system is transformed by the introduction of new units defined by equations (13), and, if we denote by Δ'_1, Δ'_2, ∇' respectively the results of replacing $A_1, A_2, \ldots\ldots, A_m$ in these determinants by $A'_1, A'_2, \ldots\ldots, A'_m$ respectively, then, if τ denotes the determinant of the substitution, we have

$$\Delta'_1 = \tau^2\Delta_1, \quad \Delta'_2 = \tau^2\Delta_2, \quad \nabla' = \tau^2\nabla.$$

The *nullity* of Δ_1 is zero if $\Delta_1 \neq 0$; and the nullity of Δ_1 is equal to p if the minors of Δ_1 of order $n-p+1$ are all zero, but not all the minors of order $n-p$.

A subsystem $(B_1, B_2, \ldots\ldots, B_{m'})$ of the system $(A_1, A_2, \ldots\ldots, A_m)$ is *invariant* to the latter if

$$A_iB_j = c'_{ij1}B_1 + \ldots\ldots c'_{ijm'}B_{m'}$$
$$B_jA_i = c''_{ij1}B_1 + \ldots\ldots c''_{ijm'}B_{m'} \qquad (i=1, 2, \ldots\ldots, m;\ j=1, 2, \ldots\ldots, m').$$

The subsystem $(B_1, B_2, \ldots\ldots, B_m)$ is *nilpotent* if it contains no idempotent number, in which case, every number of the system is nilpotent. A *maximal invariant nilpotent subsystem* of $(A_1, A_2, \ldots\ldots, A_m)$ contains every invariant nilpotent subsystem of $(A_1, A_2, \ldots\ldots, A_m)$.

A system termed by Cartan is *simple* if it contains no invariant subsystem, and *semi-simple* if its invariant subsystems are simple. See following note.

by $\varkappa (1 \leqq \varkappa \leqq \bar{r})$ mutually nilfactorial quadrates of $s_1^2, s_2^2, \ldots, s_\kappa^2$ units respectively, where

$$m = \Sigma_{p=1}^{\kappa} s_p^2, \quad \bar{r} = \Sigma_{p=1}^{\kappa} s_p . \tag{33}$$

That is to say, we can now find m linearly independent numbers

$$I_{uv}^{(p)} = \Sigma_{i=1}^{m} \tau_{uvi}^{(p)} A_i, \qquad (p = 1, 2, \ldots, \varkappa;\ u, v = 1, 2, \ldots, s_p) \tag{34}$$

of the number system $(A_1, A_2, \ldots, A_m)$ such that

$$I_{uw}^{(p)} I_{wv}^{(p)} = I_{uv}^{(p)}, \quad I_{uw}^{(p)} I_{w'v}^{(p)} = 0,$$
$$(p = 1, 2, \ldots, \varkappa;\ u, w, w', v = 1, 2, \ldots, s_p;\ w' \neq w), \tag{35}$$

$$I_{uv}^{(p)} I_{u'v'}^{(q)} = 0,$$
$$(p, q = 1, 2, \ldots, \varkappa;\ q \neq p;\ u, v = 1, 2, \ldots, s_p;\ u', v' = 1, 2, \ldots, s_q); \tag{36}$$

and these m numbers we can now take as a new system of units.[*] We shall then have

$$A_i = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} g_{puv}^{(i)} I_{uv}^{(p)}, \qquad (i = 1, 2, \ldots, m). \tag{37}$$

Further, for any transformation $X = \Sigma_{i=1}^{m} x_i A_i$ of G,

$$X = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} x_{uv}^{(p)} I_{uv}^{(p)}, \tag{38}$$

where

$$x_{uv}^{(p)} = \Sigma_{i=1}^{m} x_i g_{puv}^{(i)}, \qquad (p = 1, 2, \ldots, \varkappa;\ u, v = 1, 2, \ldots, s_p); \tag{39}$$

and, conversely, the transformation X determined by equation (38), for any system of scalars

$$x_{uv}^{(p)}, \qquad (p = 1, 2, \ldots, \varkappa;\ u, v = 1, 2, \ldots, s_p),$$

belongs to G in consequence of (34) and what was stated on p. 337. Finally, since 1 is a transformation of G, we have

$$1 = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} g_{puv}^{(0)} I_{uv}^{(p)};$$

and, therefore, by (35) and (36),

$$I_{u'u'}^{(p')} = I_{u'u'}^{(p')} 1 I_{u'u'}^{(p')} = g_{p'u'u'}^{(0)} I_{u'u'}^{(p')}$$
$$0 = I_{u'u'}^{(p')} 1 I_{v'v'}^{(p')} = g_{p'u'v'}^{(0)} I_{u'v'}^{(p')}, \quad (p' = 1, 2, \ldots, \varkappa;\ u', v' = 1, 2, \ldots, s_{p'};\ v' \neq u'):$$

whence follows

$$g_{p'u'u'}^{(0)} = 1, \quad g_{p'u'v'}^{(0)} = 0, \qquad (p' = 1, 2, \ldots, \varkappa;\ u', v' = 1, 2, \ldots, s_{p'};\ v' \neq u');$$

and thus

$$1 = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} I_{uu}^{(p)} . \tag{40}$$

---

[*] See paper referred to above, pp. 647–649. See also note to p. 341 of the present paper where it is shown that $\bar{r}$ is equal to the number of distinct roots of $\Omega(\rho) = 0$. Cartan has shown that a hypercomplex number system containing no invariant nilpotent subsystem is constituted by simple systems (quadrates) mutually nilfactorial, that is, such that the product is zero of any number of any one constituent simple system by any number of any other of the constituent simple systems. See *Comptes Rendus*, Vol. CXXIV (1897), p. 1218.

From equations (35) and (36), it now follows, by *Theorem 1*, that

$$mS_1 I^{(p)}_{uu} = s_p = mS_2 I^{(p)}_{uu}, \qquad (p=1, 2, \ldots, \varkappa;\ u=1, 2, \ldots, s_p).^* \tag{41}$$

From equation (35), we derive

$$\bar{S} I^{(p)}_{uu} = \bar{S} I^{(p)}_{uv} I^{(p)}_{vu} = \bar{S} I^{(p)}_{vu} I^{(p)}_{uv} = \bar{S} I^{(p)}_{vv}, \qquad (p=1, 2, \ldots, \varkappa;\ u, v=1, 2, \ldots, s_p). \tag{42}$$

Whence, if we write

$$\sigma_p = n \bar{S} I^{(p)}_{uu}, \qquad (p=1, 2, \ldots, \varkappa;\ u=1, 2, \ldots, s_p) \tag{43}$$

and

$$n_p = s_p \sigma_p, \qquad (p=1, 2, \ldots, \varkappa), \tag{44}$$

we shall have

$$n = n \bar{S} 1 = \Sigma^{\kappa}_{p=1} \Sigma^{s_p}_{u=1} n \bar{S} I^{(p)}_{uu} = \Sigma^{\kappa}_{p=1} s_p \sigma_p = \Sigma^{\kappa}_{p=1} n_p, \tag{45}$$

in consequence of (40).

By (35), $I^{(p)}_{uu}$ is idempotent for $1 \leqq p \leqq \varkappa$ and $1 \leqq u = s_p$; and thus it follows from what was stated on p. 342 that $\sigma_p = n \bar{S} I^{(p)}_{uu}$ is the multiplicity of the root 1 of the characteristic equation of $I^{(p)}_{uu}$. Whence, if

$$J^{(p)}_{uu} = \Sigma^{N_{p-1}+u\sigma_p}_{\mu=N_{p-1}+\overline{u-1}\sigma_p+1} T_{\mu\mu}, \qquad (p=1, 2, \ldots, \varkappa;\ u=1, 2, \ldots, s_p), \tag{46}$$

where

$$N_0 = 0,\ N_p = s_1\sigma_1 + s_2\sigma_2 + \ldots + s_p\sigma_p = n_1 + n_2 + \ldots + n_p, \qquad (p=1, 2, \ldots, \varkappa), \tag{47}$$

it follows, by (35), (36), and (40), that a linear homogeneous transformation ω of non-zero determinant in the variables $\xi_1, \xi_2, \ldots, \xi_n$ and $\xi'_1, \xi'_2, \ldots, \xi'_n$ can be found such that

$$I^{(p)}_{uu} = \omega J^{(p)}_{uu} \omega^{-1}, \qquad (p=1, 2, \ldots, \varkappa;\ u=1, 2, \ldots, s_p).\dagger \tag{48}$$

---

* For if $X = \Sigma^{\kappa}_{q=1} \Sigma^{s_q}_{u'=1} \Sigma^{s_q}_{v'=1} x^{(q)}_{u'v'} I^{(q)}_{u'v'}$ is any transformation of G satisfying the equation $I^{(p)}_{uu} X = X$, then $x^{(q)}_{u'v'} = 0$ for $q \neq p$, and $x^{(p)}_{u'v'} = 0$ for $u' \neq u$. Therefore, there is a system of just s_p linearly independent transformations of G satisfying this equation, namely, $I^{(p)}_{uv'}$ for $v' = 1, 2, \ldots, s_p$; and, in terms of these transformations, every transformation of G satisfying the above equation can be expressed linearly. Similarly, $I^{(p)}_{u'u}$ for $u' = 1, 2, \ldots, s_p$ constitute a system of linearly independent transformations of G satisfying the equation $XI^{(p)}_{uu} = X$, and any transformation of G satisfying this equation can be expressed linearly in terms of these transformations.

† Thus, let

$$A_0 = \Sigma^{\kappa}_{p=1} \Sigma^{s_p}_{u=1} \lambda^{(p)}_u I^{(p)}_{uu},$$

where the λ's are distinct. The λ's are then the roots of the characteristic equation of A_0; $\lambda^{(p)}_u$, for $1 \leqq p \leqq \kappa$ and $1 \leqq u \leqq s_p$, being of multiplicity σ_p. For, by (35) and (36),

$$A^{\mu}_0 = \Sigma^{\kappa}_{p=1} \Sigma^{s_p}_{u=1} (\lambda^{(p)}_u)^{\mu} I^{(p)}_{uu}$$

for any integer μ; and, therefore, by (43),

$$nSA^{\mu}_0 = \Sigma^{\kappa}_{p=1} \Sigma^{s_p}_{u=1} (\lambda^{(p)}_u)^{\mu} nSI^{(p)}_{uu} = \Sigma^{\kappa}_{p=1} \Sigma^{s_p}_{u=1} \sigma_p (\lambda^{(p)}_u)^{\mu}.$$

But nSA^{μ}_0, see p. 342, is the sum of the roots of the characteristic equation of A^{μ}_0; and, therefore, is the

Let now

$$J_{uv}^{(p)}=\omega^{-1}I_{uv}^{(p)}\omega, \qquad (p=1,2,\ldots\ldots,\varkappa;\ u,v=1,2,\ldots\ldots,s_p;\ v\neq u). \tag{49}$$

Then, by (35), (36), and (48),

$$\begin{gathered}J_{uw}^{(p)}J_{wv}^{(p)}=\omega^{-1}I_{uw}^{(p)}\omega\cdot\omega^{-1}I_{wv}^{(p)}\omega=\omega^{-1}I_{uv}^{(p)}\omega=J_{uv}^{(p)},\\ J_{uw}^{(p)}J_{w'v}^{(p)}=\omega^{-1}I_{uw}^{(p)}\omega\cdot\omega^{-1}I_{w'v}^{(p)}\omega=0,\\ (p=1,2,\ldots\ldots,\varkappa;\ u,w,w',v=1,2,\ldots\ldots,s_p;\ w'\neq w),\end{gathered} \tag{50}$$

$$\begin{gathered}J_{uv}^{(p)}J_{u'v'}^{(q)}=\omega^{-1}I_{uv}^{(p)}\omega\cdot\omega^{-1}I_{u'v'}^{(q)}\omega=0,\\ (p,q=1,2,\ldots\ldots,\varkappa;\ q\neq p;\ u,v=1,2,\ldots\ldots,s_p;\ u'\,v'=1,2,\ldots\ldots,s_q);\end{gathered} \tag{51}$$

and, therefore, if

$$J^{(p)}=\Sigma_{u=1}^{s_p}J_{uu}^{(p)}=\Sigma_{\mu=1}^{n_p}T_{N_{p-1}+\mu,\,N_{p-1}+\mu}, \qquad (p=1,2,\ldots\ldots,\varkappa), \tag{52}$$

we then have

$$J_{uv}^{(p)}J^{(q)}=0=J^{(q)}J_{uv}^{(p)}, \quad (p,q=1,2,\ldots\ldots,\varkappa;\ q\neq p;\ u,v=1,2,\ldots\ldots,s_p). \tag{53}$$

Whence follows

$$J_{uv}^{(p)}=\Sigma_{\mu=1}^{n_p}\Sigma_{\nu=1}^{n_p}\theta_{uv\mu\nu}^{(p)}T_{N_{p-1}+\mu,\,N_{p-1}+\nu}, \quad (p=1,2,\ldots\ldots,\varkappa;\ u,v=1,2,\ldots\ldots,s_p), \tag{54}$$

where, for $p=1,2,\ldots\ldots,\varkappa$ and $u=1,2,\ldots\ldots,s_p$,

$$\begin{gathered}\theta_{u,u,\overline{u-1}\,\sigma_p+\mu,\,\overline{u-1}\,\sigma_p+\mu}^{(p)}=1,\ \theta_{u,u,w\,\sigma_p+\mu,\,w\,\sigma_p+\mu}^{(p)}=0,\\ (w=0,1,\ldots\ldots,\overline{u-2},u,\ldots\ldots,s_p-1;\ \mu=1,2,\ldots\ldots,\sigma_p)\end{gathered} \tag{55}$$

$$\theta_{uu\mu\nu}^{(p)}=0, \qquad (\mu,\nu=1,2,\ldots\ldots,n_p;\ \nu\neq\mu) \tag{56}$$

by (46). Wherefore, if, for any system of m scalars $x_{uv}^{(p)}$, $(p=1,2,\ldots\ldots,\varkappa;$ $u,v=1,2,\ldots\ldots,s_p)$,

$$\Theta_p=\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}x_{uv}^{(p)}\omega^{-1}I_{uv}^{(p)}\omega=\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}x_{uv}^{(p)}J_{uv}^{(p)}, \qquad (p=1,2,\ldots\ldots,\varkappa), \tag{57}$$

---

sum of the μ-th powers of the roots of the characteristic equation of A_0. Further, by (40),

$$A_0-\lambda_u^{(p)}\,(1\leqq p\leqq\kappa;\ 1\leqq u\leqq s_p)$$

is linear in the I's, the coefficient of $I_{uu}^{(p)}$ being zero; and thus

$$\psi(A_0)=(A_0-\lambda_1^{(1)})\ldots\ldots(A_0-\lambda_{s_1}^{(1)})\ldots\ldots(A_0-\lambda_1^{(\kappa)})\ldots\ldots(A_0-\lambda_{s_\kappa}^{(\kappa)})=0$$

by (35) and (36). Therefore, the nullity of $A_0-\lambda_u^{(p)}$ is equal to σ_p, the multiplicity of $\lambda_u^{(p)}$. Whence it follows that a matrix ω of non-zero determinant can be found such that

$$A_0=\omega J\omega^{-1},$$

where

$$J=\Sigma_{p=1}^{\kappa}\Sigma_{u=1}^{s_p}\Sigma_{\mu=N_{p-1}+\overline{u-1}\sigma_p+1}^{N_{p-1}+u\sigma_p}\lambda_u^{(p)}T_{\mu\mu},$$

and N_p is given by (47). If now

$$\chi_u^{(p)}(\rho)\equiv\frac{\psi(\rho)}{\rho-\lambda_u^{(p)}},$$

$$f_u^{(p)}(\rho)\equiv\chi_u^{(p)}(\rho)/\chi_u^{(p)}(\lambda_u^{(p)}),$$

then

$$I_{uu}^{(p)}=f_u^{(p)}(A_0)=f_u^{(p)}(\omega J\omega^{-1})=\omega f_u^{(p)}(J)\omega^{-1}=\omega J_{uu}^{(p)}\omega^{-1},$$

for $p=1,2,\ldots\ldots,\kappa$ and $u=1,2,\ldots\ldots,s_p$, where J_{uu} is given by (46).

it then follows that

$$\Theta_p = \Sigma_{\mu=1}^{n_p} \Sigma_{\nu=1}^{n_p} \vartheta_{\mu\nu}^{(p)} T_{N_{p-1}+\mu,\, N_{p-1}+\nu}, \qquad (p=1, 2, \ldots, \varkappa), \tag{58}$$

where

$$\vartheta_{\mu\nu}^{(p)} = \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} x_{uv}^{(p)} \theta_{uv\mu\nu}^{(p)}, \qquad (p=1, 2, \ldots, \varkappa;\ \mu, \nu=1, 2, \ldots, n_p). \tag{59}$$

Whence, if

$$X = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} x_{uv}^{(p)} I_{uv}^{(p)} \tag{60}$$

is the general transformation of G, and

$$\Theta = \omega^{-1} X \omega, \tag{61}$$

it follows that

$$\Theta = \Sigma_{p=1}^{\kappa} \Theta_p = \Sigma_{p=1}^{\kappa} \Sigma_{\mu=1}^{n_p} \Sigma_{\nu=1}^{n_p} \vartheta_{\mu\nu}^{(p)} T_{N_{p-1}+\mu,\, N_{p-1}+\nu}; \tag{62}$$

that is to say, $\Theta = \omega^{-1} X \omega$ is represented by the matrix

$$\begin{vmatrix} \Theta_1, & 0, & \ldots, & 0 \\ 0, & \Theta_2, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & \Theta_\kappa \end{vmatrix}$$

in which Θ_p $(p=1, 2, \ldots, \varkappa)$ represents the square array

$$\begin{vmatrix} \vartheta_{11}^{(p)}, & \vartheta_{12}^{(p)}, & \ldots, & \vartheta_{1n_p}^{(p)} \\ \vartheta_{21}^{(p)}, & \vartheta_{22}^{(p)}, & \ldots, & \vartheta_{2n_p}^{(p)} \\ \ldots & \ldots & \ldots & \ldots \\ \vartheta_{n_p1}^{(p)}, & \vartheta_{n_p2}^{(p)}, & \ldots, & \vartheta_{n_pn_p}^{(p)} \end{vmatrix}$$

with n_p^2 constituents, and the zero's represent rectangular arrays whose constituents are all zero. In other words, if

$$(\xi_1', \xi_2', \ldots, \xi_n') = \Theta(\xi_1, \xi_2, \ldots, \xi_n), \tag{63}$$

then

$$\begin{aligned} &(\xi'_{N_{p-1}+1}, \xi'_{N_{p-1}+2}, \ldots, \xi'_{N_p}) \\ &\quad = \begin{pmatrix} \vartheta_{11}^{(p)}, & \vartheta_{12}^{(p)}, & \ldots, & \vartheta_{1n_p}^{(p)} \\ \vartheta_{21}^{(p)}, & \vartheta_{22}^{(p)}, & \ldots, & \vartheta_{2n_p}^{(p)} \\ \ldots & \ldots & \ldots & \ldots \\ \vartheta_{n_p1}^{(p)}, & \vartheta_{n_p2}^{(p)}, & \ldots, & \vartheta_{n_pn_p}^{(p)} \end{pmatrix} (\xi_{N_{p-1}+1}, \xi_{N_{p-1}+2}, \ldots, \xi_{N_p}) \end{aligned} \qquad (p=1, 2, \ldots, \varkappa). \tag{64}$$

For the further determination of the matrices $\Theta_1, \Theta_2, \ldots, \Theta_\kappa$, we may proceed as follows: Let $\varkappa=1$, in which case

$$m = s_1^2, \quad n = s_1\sigma_1 = n_1 = N_1,$$

by (33), (44), (45), and (47). In the American Journal of Mathematics, Vol. XII (1899), p. 391, I have shown that a quadrate of order $n=n'n''$ is the product of two quadrates of order n' and n'' respectively, or, what is the same thing, that a matrix of order $n=n'n''$ may be regarded as a matrix of order n'

whose constituents are matrices of order n''.* Whence, if T denotes the general linear homogeneous transformation in $n_1=s_1\sigma_1$ variables, it is represented by the matrix

$$\left|\begin{matrix} M_{11}, & M_{12}, & \ldots, & M_{1s_1} \\ M_{21}, & M_{22}, & \ldots, & M_{2s_1} \\ \ldots & \ldots & \ldots & \ldots \\ M_{s_1}, & M_{s_2}, & \ldots, & M_{s_1 s_1} \end{matrix}\right|,$$

where the M's are matrices of order σ_1. Or, otherwise expressed, let $\bar{T}_{uv}$, for $u, v=1, 2, \ldots, s_1$, denote the linear transformation

$$\bar{\xi}'_1=0, \ldots, \bar{\xi}'_{u-1}=0,\ \xi'_u=\bar{\xi}_v,\ \xi'_{u+1}=0, \ldots, \xi'_{s_1}=0 \qquad (65)$$

in s_1 variables; and let $\bar{\bar{T}}_{\mu\nu}$, for $\mu, \nu=1, 2, \ldots, \sigma_1$, denote the linear transformation

$$\bar{\bar{\xi}}'_1=0, \ldots, \bar{\bar{\xi}}'_{\mu-1}=0,\ \bar{\bar{\xi}}'_\mu=\bar{\bar{\xi}}_\nu,\ \bar{\bar{\xi}}'_{\mu+1}=0, \ldots, \bar{\bar{\xi}}'_{\sigma_1}=0 \qquad (66)$$

in σ_1 variables. Then

$$\bar{T}_{uw}\bar{T}_{wv}=\bar{T}_{uv},\ \bar{T}_{uw}\bar{T}_{w'v}=0, \qquad (u, w, w', v=1, 2, \ldots, s_1;\ w' \mp w), \qquad (67)$$

$$1=\bar{T}_{11}+\bar{T}_{22}+\ldots+\bar{T}_{s_1 s_1}, \qquad (68)$$

$$\bar{\bar{T}}_{\mu\lambda}\bar{\bar{T}}_{\lambda\nu}=\bar{\bar{T}}_{\mu\nu},\ \bar{\bar{T}}_{\mu\lambda}\bar{\bar{T}}_{\lambda'\nu}=0, \qquad (\mu, \lambda, \lambda', \nu=1, 2, \ldots, \sigma_1;\ \lambda' \mp \lambda), \qquad (69)$$

$$1=\bar{\bar{T}}_{11}+\bar{\bar{T}}_{22}+\ldots+\bar{\bar{T}}_{\sigma_1\sigma_1}, \qquad (70)$$

corresponding to (22) and (19). If now we assume that

$$\bar{T}_{uv}\bar{\bar{T}}_{\mu\nu}=\bar{\bar{T}}_{\mu\nu}\bar{T}_{uv}, \qquad (u, v=1, 2, \ldots, s_1;\ \mu, \nu=1, 2, \ldots, \sigma_1), \qquad (71)$$

then, for any two integers μ', ν' from 1 to $n_1=s_1\sigma_1$, we may put

$$T_{\mu'\nu'}=\bar{T}_{uv}\bar{\bar{T}}_{\mu\nu}, \qquad (72)$$

where μ, ν are the smallest positive residues of μ', ν' respectively *modulo* σ_1, and u, v are determined by the equations

$$\mu'=(u-1)\sigma_1+\mu, \quad \nu'=(v-1)\sigma_1+\nu. \qquad (73)$$

For the units $T_{\mu'\nu'}$ will then have the required multiplication table of the quadrate of order $n=n_1=s_1\sigma_1$ given by (22). The general number T of this quadrate, that is, the general linear homogeneous transformation in $n_1=s_1\sigma_1$ variables, is now given by the equation

$$\begin{aligned} T=\Sigma_{\mu=1}^{n_1}\Sigma_{\nu=1}^{n_1} c_{\mu\nu}T_{\mu\nu} &= \Sigma_{u,\,v=1}^{s_1}\Sigma_{\mu,\,\nu=1}^{\sigma_1} c_{\overline{u-1}\,\sigma_1+\mu,\ \overline{v-1}\,\sigma_1+\nu}\,\bar{T}_{uv}\bar{\bar{\mathrm{T}}}_{\mu\nu} \\ &= \Sigma_{u=1}^{s_1}\Sigma_{v=1}^{s_1} M_{uv}\bar{T}_{uv}=\Sigma_{u=1}^{s_1}\Sigma_{v=1}^{s_1}\bar{T}_{uv}M_{uv}, \end{aligned} \qquad (74)$$

where

$$M_{uv}=\Sigma_{\mu=1}^{\sigma_1}\Sigma_{\nu=1}^{\sigma_1} c_{\overline{u-1}\,\sigma_1+\mu,\ \overline{v-1}\,\sigma_1+\nu}\,\bar{\bar{T}}_{\mu\nu}, \qquad (u, v=1, 2, \ldots, s_1).\dagger \qquad (75)$$

---

* This theorem was obtained independently by E. Study. See "Math. papers read at the Internat. Math. Congress, Chicago, 1893", p. 378.

† Two matrices $\Sigma_{u=1}^{s_1}\Sigma_{v=1}^{s_1} M_{uv}\bar{T}_{uv}$ and $\Sigma_{u=1}^{s_1}\Sigma_{v=1}^{s_1} M_{uv}^{(1)}\bar{T}_{uv}$ are equal if, and only if, $M_{uv}^{(1)}=M_{uv}$ for $u, v=1, 2, \ldots, s_1$.

In particular, by (46) and (70),

$$J^{(1)}_{uu}=\Sigma^{\sigma_1}_{\mu=1}\, T_{\overline{u-1}\,\sigma_1+\mu,\, \overline{u-1}\,\sigma_1+\mu}=\Sigma^{\sigma_1}_{\mu=1}\, \overline{T}_{uu}\, \overline{\overline{T}}_{\mu\mu}=\overline{T}_{uu}, \qquad (u=1, 2, \ldots, s_1). \tag{76}$$

Let

$$J^{(1)}_{uv}=\Sigma^{s_1}_{u'=1}\Sigma^{s_1}_{v'=1}\, M^{(u,v)}_{u'v'}\, \overline{T}_{u'v'}, \qquad (u, v=1, 2, \ldots, s_1;\ v\neq u).$$

From (50), (67), (71), and (76), it follows that

$$J^{(1)}_{uv}=J^{(1)}_{uu}\, J^{(1)}_{uv}\, J^{(1)}_{vv}=\overline{T}_{uu}\, \Sigma^{s_1}_{u'=1}\Sigma^{s_1}_{v'=1}\, M^{(u,v)}_{u'v'}\, \overline{T}_{u'v'}\, \overline{T}_{vv}=M^{(u,v)}_{uv}\, \overline{T}_{uv}, \qquad (u, v=1, 2, \ldots, s_1);$$

and thus

$$M^{(u,v)}_{u'v'}=0,^{*} \qquad (u, v, u', v'=1, 2, \ldots, s_1;\ u'\neq u;\ \text{or } v'\neq v).$$

We may, therefore, delete the indices and write simply

$$J^{(1)}_{uv}=M'_{uv}\, \overline{T}_{uv}, \qquad (u, v=1, 2, \ldots, s_1), \tag{77}$$

where M'_{uv} is a matrix of order σ_1; when, by (76),

$$M'_{uu}=1, \qquad (u=1, 2, \ldots, s_1). \tag{78}$$

From (77), it follows, by (50), (67), and (71), that

$$\begin{aligned} M'_{uw}\, M'_{wv}\, \overline{T}_{uv}&=M'_{uw}\, \overline{T}_{uw}\cdot M'_{wv}\, \overline{T}_{wv}\\ &=J^{(1)}_{uw}\, J^{(1)}_{wv}\\ &=J^{(1)}_{uv}=M'_{uv}\, \overline{T}_{uv},\\ &\qquad (u, v, w=1, 2, \ldots, s_1); \end{aligned}$$

and, therefore,

$$M'_{uw}\, M'_{wv}=M'_{uv}, \qquad (u, v, w=1, 2, \ldots, s_1). \tag{79}$$

In particular, by (78),

$$M'_{uv}\, M'_{vu}=M'_{uu}=1, \qquad (u, v=1, 2 \ldots, s_1). \tag{80}$$

Let

$$\overline{\omega}=M'_{11}\, \overline{T}_{11}+M'_{12}\, \overline{T}_{22}+\ldots+M'_{1s_1}\, \overline{T}_{s_1 s_1}. \tag{81}$$

Then, by (67), (68), (71), and (80),

$$\overline{\omega}^{-1}=M'_{11}\, \overline{T}_{11}+M'_{21}\, \overline{T}_{22}+\ldots+M'_{s_1 1}\, \overline{T}_{s_1 s_1};^{\dagger} \tag{82}$$

and, therefore, by (67), (71), and (76),

$$\overline{\omega}\, J^{(1)}_{uu}=\overline{\omega}\, \overline{T}_{uu}=M'_{1u}\, \overline{T}_{uu}, \quad J^{(1)}_{uu}\, \overline{\omega}^{-1}=\overline{T}_{uu}\, \overline{\omega}^{-1}=M'_{u1}\, \overline{T}_{uu}, \qquad (u=1, 2, \ldots, s_1). \tag{83}$$

Whence follows

$$\begin{aligned} \overline{\omega}\, J^{(1)}_{uv}\, \overline{\omega}^{-1}&=\overline{\omega}\, J^{(1)}_{uu}\cdot J^{(1)}_{uv}\cdot J^{(1)}_{vv}\, \overline{\omega}^{-1}\\ &=M'_{1u}\, \overline{T}_{uu}\cdot M'_{uv}\, \overline{T}_{uv}\cdot M'_{v1}\, \overline{T}_{vv}\\ &=M'_{1u}\, M'_{uv}\, M'_{v1}\, \overline{T}_{uu}\, \overline{T}_{uv}\, \overline{T}_{vv}=\overline{T}_{uv},\\ &\qquad (u, v=1, 2, \ldots, s_1), \end{aligned} \tag{84}$$

---

* See preceding note.

† That is, if $\overline{\omega}$ is defined by (81), and $\overline{\omega}^{(1)}=\Sigma^{s_1}_{u=1} M'_{u1}\overline{T}_{uu}$, then $\overline{\omega}\,\overline{\omega}^{(1)}=1$; and therefore, $\overline{\omega}^{(1)}=\overline{\omega}^{-1}$.

by (50), (67), (71), (77), (78), and (79). Wherefore, if

$$\Theta_1 = \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} x_{uv}^{(1)} J_{uv}^{(1)} , \tag{85}$$

then, by (70) and (72),

$$\begin{aligned} \overline{\omega}\, \Theta_1 \overline{\omega}^{-1} &= \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} x_{uv}^{(1)} \overline{\omega}\, J_{uv}^{(1)} \overline{\omega}^{-1} \\ &= \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} x_{uv}^{(1)} \overline{T}_{uv} \\ &= \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} \Sigma_{\mu=1}^{\sigma_1} x_{uv}^{(1)} \overline{T}_{uv} \overline{\overline{T}}_{\mu\mu} \\ &= \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} x_{uv}^{(1)} \Sigma_{\mu=1}^{\sigma_1} T_{\overline{u-1}\,\sigma_1+\mu,\ \overline{v-1}\,\sigma_1+\mu} \, ; \end{aligned} \tag{86}$$

that is to say, $\overline{\omega}\Theta_1 \overline{\omega}^{-1}$ is represented by the matrix

$$\begin{vmatrix} x_{11}^{(1)} L, & x_{12}^{(1)} L, & \ldots\ldots, & x_{1s_1}^{(1)} L \\ x_{21}^{(1)} L, & x_{22}^{(1)} L, & \ldots\ldots, & x_{2s_1}^{(1)} L \\ \ldots & \ldots & \ldots & \ldots \\ x_{s_1 1}^{(1)} L, & x_{s_1 2}^{(1)} L, & \ldots\ldots, & x_{s_1 s_1}^{(1)} L \end{vmatrix},$$

where L is a matrix of order σ_1 whose constituents are all zero, except those in the principal diagonal which are all equal to unity.

By the composition with $\overline{\omega}$ of an interchange of the subscripts of the $N_1 = n_1 = s_1 \sigma_1$ variables $\xi_1, \xi_2, \ldots\ldots, \xi_{N_1}$, and, therefore, by a properly chosen linear substitution $\overline{\omega}_1$ with constant coefficients in these variables, we shall have

$$\overline{\omega}_1 \Theta_1 \overline{\omega}_1^{-1} = \Sigma_{u=1}^{s_1} \Sigma_{v=1}^{s_1} \Sigma_{\mu=1}^{\sigma_1} x_{uv}^{(1)} T_{\overline{\mu-1}\,s_1+u,\ (\mu-1)s_1+v} , \tag{87}$$

in which case $\overline{\omega}_1 \Theta \overline{\omega}_1^{-1}$ is represented by the matrix

$$\Psi^{(1)} = \begin{vmatrix} \Psi_1^{(1)}, & 0, & \ldots\ldots, & 0 \\ 0, & \Psi_2^{(1)}, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots\ldots, & \Psi_{\sigma_1}^{(1)} \end{vmatrix}, \tag{88}$$

where the zeros represent rectangular arrays whose constituents are all zero, and

$$\Psi_1^{(1)} = \Psi_2^{(1)} = \ldots\ldots = \Psi_{\sigma_1}^{(1)} = \begin{vmatrix} x_{11}^{(1)}, & \ldots\ldots, & x_{1s_1}^{(1)} \\ \ldots & \ldots & \ldots \\ x_{s_1 1}^{(1)}, & \ldots\ldots, & x_{s_1 s_1}^{(1)} \end{vmatrix}. \tag{89}$$

That is to say, if

$$(\xi_1', \xi_2', \ldots\ldots, \xi_{N_1}') = \overline{\omega}_1 \Theta_1 \overline{\omega}_1^{-1} (\xi_1, \xi_2, \ldots\ldots, \xi_{N_1}), \tag{90}$$

then

$$\begin{aligned} &(\xi'_{(\mu-1)s_1+1}, \xi'_{(\mu-1)s_1+2}, \ldots\ldots, \xi'_{\mu s_1}) \\ &\qquad = \left(\begin{array}{cccc} x_{11}^{(1)}, & x_{12}^{(1)}, & \ldots\ldots, & x_{1s_1}^{(1)} \\ x_{21}^{(1)}, & x_{22}^{(1)}, & \ldots\ldots, & x_{2s_1}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ x_{s_1 1}^{(1)}, & x_{s_1 2}^{(1)}, & \ldots\ldots, & x_{s_1 s_1}^{(1)} \end{array} \right) (\xi_{(\mu-1)s_1+1}, \xi_{(\mu-1)s_1+2}, \ldots\ldots, \xi_{\mu s_1}), \\ &\qquad\qquad (\mu = 1, 2, \ldots\ldots, \sigma_1). \end{aligned} \tag{91}$$

In particular, when $\varkappa=1$ (and, as assumed above $\Delta_1=\Delta_2\neq 0$), if $\sigma_1=1$, then

$$m=s_1^2=n_1^2, \quad n=s_1=n_1=N_1.$$

In this case,

$$\overline{\omega}_1\Theta_1\overline{\omega}_1^{-1}(\xi_1, \xi_2, \ldots, \xi_n)=\left|\begin{array}{l} x_{11}^{(1)}, x_{12}^{(1)}, \ldots, x_{1n}^{(1)} \\ x_{21}^{(1)}, x_{22}^{(1)}, \ldots, x_{2n}^{(1)} \\ \ldots\ldots\ldots\ldots\ldots \\ x_{n1}^{(1)}, x_{n2}^{(1)}, \ldots, x_{nn}^{(1)} \end{array}\right|(\xi_1, \xi_2, \ldots, \xi_n).$$

Wherefore, by (60), (61), and (62), if

$$\omega_0=\omega\,\overline{\omega}_1^{-1},$$

the general transformation X of G is given by the equation

$$X=\omega\,\Theta\,\omega^{-1}=\omega\,\Theta_1\,\omega^{-1}=\omega_0\left|\begin{array}{l} x_{11}^{(1)}, x_{12}^{(1)}, \ldots, x_{1n}^{(1)} \\ x_{21}^{(1)}, x_{22}^{(1)}, \ldots, x_{2n}^{(1)} \\ \ldots\ldots\ldots\ldots\ldots \\ x_{n1}^{(1)}, x_{n2}^{(1)}, \ldots, x_{nn}^{(1)} \end{array}\right|\omega_0^{-1};$$

and, since the x's are arbitrary, there is no flat through the origin invariant to G, which is, therefore, irreducible.[*] Conversely, if G is irreducible, then $\Delta_1=\Delta_2\neq 0$. For, in this case, by Burnside's theorem,[†] G contains n^2 linearly independent transformations; and thus $m=n^2$. But since $A_1, A_2, \ldots, A_m$, the maximum number of linearly independent transformations of G are, by (20), expressible linearly in terms of the $m=n^2$ matrices $T_{\mu\nu}(\mu, \nu=1, 2, \ldots, n)$, it then follows that the latter are linearly expressible in terms of the A's: therefore, the number system $(A_1, A_2, \ldots, A_m)$, being equivalent to a quadrate, can contain no invariant nilpotent subsytem; and thus $\Delta_1=\Delta_2\neq 0$.

Let now $\varkappa>1$; and for $1\leqq p\leqq\varkappa$, see (57), let

$$\Theta_p=\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}x_{uv}^{(p)}\,\omega^{-1}\,I_{uv}^{(p)}\,\omega=\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}x_{uv}^{(p)}\,J_{uv}^{(p)}. \tag{92}$$

Then, by (58), Θ_p may be regarded as a transformation in the n_p variables $\xi_{N_{p-1}+1}, \xi_{N_{p-1}+2}, \ldots, \xi_{N_p}$; and, by what has just been shown (pp. 347–350), it follows that a linear substitution $\overline{\omega}_p$ in these variables with constant coefficients can be found such, if

$$(\xi'_{N_{p-1}+1}, \xi'_{N_{p-1}+2}, \ldots, \xi'_{N_p})=\overline{\omega}_p\,\Theta_p\,\overline{\omega}_p^{-1}(\xi_{N_{p-1}+1}, \xi_{N_{p-1}+2}, \ldots, \xi_{N_p}), \tag{93}$$

---

[*] See p. 358. A subgroup of the general linear homogeneous group in n variables is irreducible if, and only if, there is a p-flat $(1\leqq p<n)$ through the origin invariant to every transformation of the subgroup.

[†] *Proc. Lond. Math. Soc.*, 2d Ser. Vol. III (1905), p. 433.

that then

$$(\xi'_{N_{p-1}+\overline{\nu-1}s_p+1},\ \xi'_{N_{p-1}+\overline{\nu-1}s_p+2},\ \ldots\ldots,\ \xi'_{N_{p-1}+\nu s_p})$$
$$=\left(\begin{array}{cccc} x_{11}^{(p)}, & x_{12}^{(p)}, & \ldots\ldots, & x_{1s_p}^{(p)} \\ x_{21}^{(p)}, & x_{22}^{(p)}, & \ldots\ldots, & x_{2s_p}^{(p)} \\ \ldots & \ldots & \ldots & \ldots \\ x_{s_p1}^{(p)}, & x_{s_p2}^{(p)}, & \ldots\ldots, & x_{s_ps_p}^{(p)} \end{array}\right)(\xi_{N_{p-1}+\overline{\nu-1}s_p+1},\ \xi_{N_{p-1}+\overline{\nu-1}s_p+2},\ \ldots\ldots,\ \xi_{N_{p-1}+\nu s_p}) \quad (\nu=1,2,\ldots\ldots,\sigma_p)\,; \tag{94}$$

that is,

$$\overline{\omega}_p\,\Theta_p\,\overline{\omega}_p^{-1}=\Sigma_{u=1}^{s_p}\,\Sigma_{v=1}^{s_p}\,\Sigma_{\nu=1}^{\sigma_p}\,x_{uv}^{(p)}\,T_{N_{p-1}+\overline{\nu-1}s_p+u,\ N_{p-1}+\overline{\nu-1}s_p+v}\,. \tag{95}$$

The $\varkappa$ sets of variables $\xi_{N_{p-1}+1}, \xi_{N_{p-1}+2}, \ldots\ldots, \xi_{N_p}$, for $p=1, 2, \ldots\ldots, \varkappa$, are mutually exclusive and comprise all the ξ's. Whence, if

$$\Theta=\Sigma_{p=1}^{\kappa}\,\Sigma_{u=1}^{s_p}\,\Sigma_{v=1}^{s_p}\,x_{uv}^{(p)}\,J_{uv}^{(p)}=\Sigma_{p=1}^{\kappa}\,\Theta_p\,, \tag{96}$$

and if

$$\Psi=\Sigma_{p=1}^{\kappa}\,\Sigma_{u=1}^{s_p}\,\Sigma_{v=1}^{s_p}\,\Sigma_{\mu=1}^{\sigma_p}\,x_{uv}^{(p)}\,T_{N_{p-1}+\overline{\mu-1}s_p+u,\ N_{p-1}+\overline{\mu-1}s_p+v}\,, \tag{97}$$

in which case

$$\Psi=\begin{vmatrix} \Psi^{(1)}, & 0, & \ldots\ldots, & 0 \\ 0, & \Psi^{(2)}, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots\ldots, & \Psi^{(\kappa)} \end{vmatrix}, \tag{98}$$

where

$$\Psi^{(p)}=\begin{vmatrix} \Psi_1^{(p)}, & 0, & \ldots\ldots, & 0 \\ 0, & \Psi_2^{(p)}, & \ldots\ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots\ldots, & \Psi_{\sigma_p}^{(p)} \end{vmatrix} \quad (p=1, 2, \ldots\ldots, \varkappa), \tag{99}$$

the zeros in this matrix (also the zeros in the preceding matrix) representing rectangular arrays whose constituents are all zero, and

$$\Psi_1^{(p)}=\Psi_2^{(p)}=\ldots\ldots\Psi_{\sigma_p}^{(p)}=\begin{vmatrix} x_{11}^{(p)}, & x_{12}^{(p)}, & \ldots\ldots, & x_{1s_p}^{(p)} \\ x_{21}^{(p)}, & x_{22}^{(p)}, & \ldots\ldots, & x_{2s_p}^{(p)} \\ \ldots & \ldots & \ldots & \ldots \\ x_{s_p1}^{(p)}, & x_{s_p2}^{(p)}, & \ldots\ldots, & x_{s_ps_p}^{(p)} \end{vmatrix} \quad (p=1, 2, \ldots\ldots, \varkappa), \tag{100}$$

it then follows that a linear substitution $\overline{\omega}_0$ in the ξ's with constant coefficients can be found such that

$$\overline{\omega}_0\,\Theta\,\overline{\omega}_0^{-1}=\Psi. \tag{101}$$

Wherefore, if

$$X=\Sigma_{p=1}^{\kappa}\,\Sigma_{u=1}^{s_p}\,\Sigma_{v=1}^{s_p}\,x_{uv}^{(p)}\,I_{uv}^{(p)}, \tag{102}$$

where the x's are any set of $m=\Sigma_{p=1}^{\kappa}s_p^2$ scalars, is any transformation of G, and

$$\omega'=\omega\,\overline{\omega}_0^{-1}, \tag{103}$$

then, by (61),

$$X=\omega\,\Theta\,\omega^{-1}=\omega'\,\Psi\,\omega'^{-1}, \tag{104}$$

where the coefficients of ω' are constant. That is to say, if

$$(\xi'_1, \xi'_2, \ldots, \xi'_n) = \omega'^{-1} X \omega' (\xi_1, \xi_2, \ldots, \xi_n), \qquad (105)$$

then

$$\begin{aligned}&(\xi'_{N_{p-1}+\overline{\mu-1}s_p+1}, \xi'_{N_{p-1}+\overline{\mu-1}s_p+2}, \ldots, \xi'_{N_{p-1}+\mu s_p}) \\ &= \begin{pmatrix} x^{(p)}_{11}, & x^{(p)}_{12}, & \ldots, & x^{(p)}_{1s_p} \\ x^{(p)}_{21}, & x^{(p)}_{22}, & \ldots, & x^{(p)}_{2s_p} \\ \cdots & \cdots & \cdots & \cdots \\ x^{(p)}_{s_p1}, & x^{(p)}_{s_p2}, & \ldots, & x^{(p)}_{s_ps_p} \end{pmatrix} (\xi_{N_{p-1}+\overline{\mu-1}s_p+1}, \xi_{N_{p-1}+\overline{\mu-1}s_p+2}, \ldots, \xi_{N_{p-1}+\mu s_p}) \\ &\qquad (p=1, 2, \ldots, \varkappa;\ \mu=1, 2, \ldots, \sigma_p). \qquad (106)\end{aligned}$$

Since the m scalars $x^{(p)}_{uv}$ $(p=1, 2, \ldots, \varkappa;\ u, v=1, 2, \ldots, s_p)$ are arbitrary, it follows that the group $G^{(p)'}_{\mu}$ defined, for $1 \leqq p \leqq \varkappa$ and $1 \leqq \mu \leqq \sigma_p$, by equations (106) is irreducible. Whence it follows that G is completely reducible.* We have, therefore, the following theorem.

THEOREM II. *If $A_1, A_2, \ldots, A_m$ is any system of m linearly independent linear homogeneous transformations in n variables $(m \leqq n^2)$ constituting a hypercomplex number system, so that*

$$A_i A_j = \Sigma^m_{k=1} \gamma_{ijk} A_k, \qquad (i, j=1, 2, \ldots, m),$$

the aggregate of numbers $\Sigma^m_{i=1} x_i A_i$ of this number system constitute a group which is completely reducible if the discriminant of either of the quadratic forms $S_1(\Sigma^m_{i=1} x_i A_i)^2$, $S_2(\Sigma^m_{i=1} x_i A_i)^2$, or $\bar{S}(\Sigma^m_{i=1} x_i A_i)^2$, is not zero (in which case the discriminant of neither of the other two forms is zero), provided the determinant is not zero of every transformation of the group.†

§ 2. *Conditions Sufficient for Similarity of Two Subgroups of the General Linear Homogeneous Group the Coefficients of whose Transformations are Linear Functions of Certain Parameters.*

Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be two systems of linearly independent linear homogeneous transformations in n variables, each constituting a hypercomplex number system, so that

$$A_i A_j = \Sigma^m_{k=1} \gamma_{ijk} A_k, \qquad (i, j=1, 2, \ldots, m), \qquad (107)$$

$$B_i B_j = \Sigma^m_{k=1} \gamma'_{ijk} B_k, \qquad (i, j=1, 2, \ldots, m); \qquad (108)$$

---

* The group G is irreducible if $\kappa=1$, $\sigma=1$. I follow Loewy in regarding the case of irreducibility as included in that of complete reducibility. See p. 359.

† Since, for $1 \leqq i \leqq m$, $1 \leqq j \leqq m$,

$$\frac{\partial^2}{\partial x_i \partial x_j} S_1(\Sigma^m_{k=1} x_k A_k)^2 = 2 S_1 A_i A_j,$$

it follows that Δ_1 is the discriminant of $S_1(\Sigma^m_{k=1} x_k A_k)^2$. Similarly, Δ_2 and ∇ are, respectively, the discriminants of $S_2(\Sigma^m_{k=1} x_k A_k)^2$ and $\bar{S}(\Sigma^m_{k=1} x_k A_k)^2$.

and let G and H, respectively, be the groups constituted by the aggregates of transformations

$$X=\Sigma_{i=1}^{m} x_i A_i, \tag{109}$$

$$Y=\Sigma_{i=1}^{m} y_i B_i, \tag{110}$$

for all systems of values of the x's and the y's.

The condition necessary and sufficient that the groups G and H shall be *similar* is that a linear substitution Ω with constant coefficients can be found such that

$$Y=\Sigma_{i=1}^{m} y_i B_i=\Sigma_{i=1}^{m} x_i \Omega A_i \Omega^{-1}=\Omega X \Omega^{-1}, \tag{111}$$

where the x's are determined as linear functions of the y's by the equations

$$y_i=\tau_{1i} x_1+\tau_{2i} x_2+\ldots\ldots+\tau_{mi} x_m, \qquad (i=1, 2, \ldots\ldots, m). \tag{112}$$

Let, first, G and H be similar. Then the number systems $(A_1, A_2, \ldots\ldots, A_m)$ and $(B_1, B_2, \ldots\ldots, B_m)$ are equivalent. For, if

$$B'_i \equiv \Sigma_{h=1}^{m} \tau_{ih} B_h=\Omega A_i \Omega^{-1}, \qquad (i=1, 2, \ldots\ldots, m), \tag{113}$$

by (111) and (112), it follows that the B''s are linearly independent and that

$$B'_i B'_j=\Sigma_{k=1}^{m} \gamma_{ijk} B'_k, \qquad (i, j=1, 2, \ldots\ldots, m). \tag{114}$$

Moreover,

$$\bar{S} B'_i=\bar{S} \Omega A_i \Omega^{-1}=\bar{S} A_i, \qquad (i=1, 2, \ldots\ldots, m). \tag{115}$$

Conversely, let now the number systems $(A_1, A_2, \ldots\ldots, A_m)$ and $(B_1, B_2, \ldots\ldots, B_m)$ be equivalent; and let $B'_1, B'_2, \ldots\ldots, B'_m$, where

$$B'_i=\tau_{i1} B_1+\tau_{i2} B_2+\ldots\ldots+\tau_{im} B_m, \qquad (i=1, 2, \ldots\ldots, m), \tag{116}$$

be m linearly independent functions of the B's such that

$$B'_i B'_j=\Sigma_{i=1}^{m} \gamma_{ijk} B_k, \qquad (i, j=1, 2, \ldots\ldots, m). \tag{117}$$

Moreover, let

$$\bar{S} B'_i=\bar{S} A_i, \qquad (i=1, 2, \ldots\ldots, m). \tag{118}$$

Then, as will be shown below, G and H are similar, provided the discriminant of the quadratic form $\Sigma_{i=1}^{m} \Sigma_{j=1}^{m} x_i x_j S A_i A_j$ is not zero; but not, in general, otherwise. For, let

$$\begin{vmatrix} \bar{S} B'_1 B'_1, \ldots\ldots, \bar{S} B'_1 B'_m \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{S} B'_m B'_1, \ldots\ldots, \bar{S} B'_m B'_m \end{vmatrix} = \begin{vmatrix} \bar{S} A_1 A_1, \ldots\ldots, \bar{S} A_1 A_m \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{S} A_m A_1, \ldots\ldots, \bar{S} A_m A_m \end{vmatrix} \neq 0. \tag{119}$$

Then the number system $(A_1, A_2, \ldots\ldots, A_m)$ is constituted by $\varkappa$ mutually nil-factorial quadrates $I_{uv}^{(p)}$ $(u, v=1, 2, \ldots\ldots, s_p)$ for $p=1, 2, \ldots\ldots, \varkappa$; that is, we can find

$$m=s_1^2+s_2^2+\ldots\ldots+s_\kappa^2, \qquad (\varkappa \geqq 1), \tag{120}$$

linearly independent numbers of this system, namely,

$$I_{uv}^{(p)}=\Sigma_{h=1}^{m}\tau_{uvh}^{(p)}A_h\,, \qquad (p=1,2,\ldots\ldots,\varkappa;\ u,v=1,2,\ldots\ldots,s_p), \tag{121}$$

(which we may take as a new system of units) such that

$$I_{uw}^{(p)}I_{wv}^{(p)}=I_{uv}^{(p)},\ I_{uw}^{(p)}I_{w'v}^{(p)}=0,$$
$$(p=1,2,\ldots\ldots,\varkappa;\ u,v,w,w'=1,2,\ldots\ldots,s;\ w'\neq w), \tag{122}$$

$$I_{uv}^{(p)}I_{u'v'}^{(q)}=0,$$
$$(p,q=1,2,\ldots\ldots,\varkappa;\ q\neq p;\ u,v=1,2,\ldots\ldots,s_p;\ u',v'=1,2,\ldots\ldots,s_q)\,; \tag{123}$$

and we shall then have

$$m\,S_1 I_{uu}^{(p)}=s_p=m\,S_2 I_{uu}^{(p)}, \qquad (p=1,2,\ldots\ldots,\varkappa;\ u=1,2,\ldots\ldots,s_p), \tag{124}$$

and

$$\bar{S}I_{11}^{(p)}=\bar{S}I_{22}^{(p)}=\ldots\ldots=\bar{S}I_{s_ps_p}^{(p)}, \qquad (p=1,2,\ldots\ldots,\varkappa).^* \tag{125}$$

For any system of scalars $x_1, x_2, \ldots\ldots, x_m$,

$$X=\Sigma_{i=1}^{m}x_iA_i=\Sigma_{p=1}^{\kappa}\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}x_{uv}^{(p)}I_{uv}^{(p)}, \tag{126}$$

where $x_{uv}^{(p)}$, for $p=1,2,\ldots\ldots,\varkappa$ and $u,v=1,2,\ldots\ldots,s_p$ are determined by the equations

$$x_i=\Sigma_{p=1}^{\kappa}\Sigma_{u=1}^{s_p}\Sigma_{v=1}^{s_p}\tau_{uvi}^{(p)}x_{uv}^{(p)}, \qquad (i=1,2,\ldots\ldots,m). \tag{127}$$

If now

$$n\bar{S}I_{uu}^{(p)}=\sigma_p\,, \qquad (p=1,2,\ldots\ldots,\varkappa;\ u,v=1,2,\ldots\ldots,s_p), \tag{128}$$

and

$$N_0=0,\ N_p=s_1\sigma_1+s_2\sigma_2+\ldots\ldots+s_p\sigma_p\,, \qquad (p=1,2,\ldots\ldots,\varkappa), \tag{129}$$

it follows, by what has been shown in § 1, that a linear substitution ω' with constant coefficients can be found such that

$$X=\omega'\Psi\omega'^{-1}, \tag{130}$$

where Ψ is defined by equations (98); that is, if

$$(\xi_1', \xi_2', \ldots\ldots, \xi_n')=\omega'^{-1}X\omega'(\xi_1, \xi_2, \ldots\ldots, \xi_n), \tag{131}$$

then

$$\begin{gathered}(\xi'_{N_{p-1}+\overline{\mu-1}s_p+1}\ \xi'_{N_{p-1}+\overline{\mu-1}s_p+2}\ \ldots\ldots\ \xi'_{N_{p-1}+\mu s_p})\\ =\begin{pmatrix} x_{11}, & x_{12}, & \ldots\ldots, & x_{1s_p}\\ x_{21}, & x_{22}, & \ldots\ldots, & x_{2s_p}\\ \ldots & \ldots & \ldots & \ldots\\ x_{s_p1}, & x_{s_p2}, & \ldots\ldots, & x_{s_ps_p}\end{pmatrix}(\xi_{N_{p-1}+\overline{\mu-1}s_p+1}\ \xi_{N_{p-1}+\overline{\mu-1}+2}\ \ldots\ldots\ \xi_{N_{p-1}+\overline{\mu-1}s_p})\\ (p=1,2,\ldots\ldots,\varkappa;\ \mu=1,2,\ldots\ldots,\sigma_p).\end{gathered} \tag{132}$$

Let

$$K_{uv}^{(p)}=\Sigma_{h=1}^{m}\tau_{uvh}^{(p)}B_h'\,, \qquad (p=1,2,\ldots\ldots,\varkappa;\ u,v=1,2,\ldots\ldots,s_p). \tag{133}$$

---

* Cf. pp. 343, 344 and 345.

Then the $m = \Sigma_{p=1}^{\kappa} s_p^2$ K's are linearly independent; and it follows from equation (117) that

$$K_{uw}^{(p)} K_{wv}^{(p)} = K_{uv}^{(p)}, \quad K_{uw}^{(p)} K_{w'v}^{(p)} = 0,$$

$$(p = 1, 2, \ldots, \varkappa; \; u, v, w, w' = 1, 2, \ldots, s_p; \; w' \neq w), \tag{134}$$

$$K_{uv}^{(p)} K_{u'v'}^{(q)} = 0,$$

$$(p, q = 1, 2, \ldots, \varkappa; \; q \neq p; \; u, v = 1, 2, \ldots, s_p; \; u', v' = 1, 2, \ldots, s_q); \tag{135}$$

and, therefore,

$$m S_1 K_{uu}^{(p)} = s_p, \qquad (p = 1, 2, \ldots, \varkappa; \; u, v = 1, 2, \ldots, s_p). \tag{136}$$

For any system of scalars $y_1, y_2, \ldots, y_m$,

$$Y = \Sigma_{i=1}^{m} y_i B_i = \Sigma_{i=1}^{m} x_i B_i' = \Sigma_{p=1}^{\kappa} \Sigma_{u=1}^{s_p} \Sigma_{v=1}^{s_p} x_{uv}^{(p)} K_{uv}^{(p)} \tag{137}$$

provided $x_1, x_2, \ldots, x_m$ are determined by the equations

$$y_i = \tau_{1i} x_1 + \tau_{2i} x_2 + \ldots + \tau_{mi} x_m, \qquad (i = 1, 2, \ldots, m), \tag{138}$$

and $x_{uv}^{(p)}$ $(p = 1, 2, \ldots, \varkappa; \; u, v = 1, 2, \ldots, s_p)$ by (127). Moreover,

$$n \bar{S} K_{uu}^{(p)} = n \Sigma_{h=1}^{m} \tau_{uuh}^{(p)} \bar{S} B_h' = n \Sigma_{h=1}^{m} \tau_{uuh}^{(p)} \bar{S} A_h = n \bar{S} I_{uu}^{(p)} = \sigma_p,$$

$$(p = 1, 2, \ldots, \varkappa; \; u = 1, 2, \ldots, s_p). \tag{139}$$

Therefore, by what was shown in § 1, a linear substitution ω'' with constant coefficients can be found such that

$$Y = \omega'' \Psi \omega''^{-1}, \tag{140}$$

where Ψ is defined by equations (98); that is, if

$$(\xi_1', \xi_2', \ldots, \xi_n') = \omega''^{-1} Y \omega'' (\xi_1, \xi_2, \ldots, \xi_n), \tag{141}$$

the ξ''s are defined by equations (132).

Whence if

$$\Omega = \omega'' \omega'^{-1}, \tag{142}$$

it follows that

$$Y = \omega'' \Psi \omega''^{-1} = \Omega \omega' \Psi \omega'^{-1} \Omega^{-1} = \Omega X \Omega^{-1}; \tag{143}$$

that is, G and H are similar. On the other hand, if

$$\begin{vmatrix} \bar{S} A_1 A_1, \ldots, \bar{S} A_1 A_m \\ \ldots\ldots\ldots\ldots \\ \bar{S} A_m A_1, \ldots, \bar{S} A_m A_m \end{vmatrix} = \begin{vmatrix} \bar{S} B_1' B_1', \ldots, \bar{S} B_1' B_m' \\ \ldots\ldots\ldots\ldots \\ \bar{S} B_m' B_1', \ldots, \bar{S} B_m' B_m' \end{vmatrix} = 0,$$

the groups G and H are not, in general, similar. Thus, let $n = 4$; and let C denote the linear homogeneous transformation

$$C(\xi_1, \xi_2, \xi_3, \xi_4) = \begin{pmatrix} 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \\ 0, 0, 0, 0 \end{pmatrix} (\xi_1, \xi_2, \xi_3, \xi_4).$$

Let

$$A_1=1,\ A_2=C^2;\quad B_1=1,\ B_2=C^3.$$

Then A_1 and A_2 are linearly independent, as are B_1 and B_2; and

$$A_1^2=A_1,\ A_1A_2=A_2=A_2A_1,\ A_2^2=0,$$
$$B_1^2=B_1,\ B_1B_2=B_2=B_2B_1,\ B_2^2=0.$$

Therefore, the number systems (A_1, A_2) and (B_1, B_2) have the same multiplication table. Moreover,

$$\bar{S}A_1=1=\bar{S}B_1,\ \bar{S}A_2=0=\bar{S}B_2.$$

But the group

$$(\xi_1', \xi_2', \xi_3', \xi_4')=\begin{vmatrix} x_1, & 0, & x_2, & 0 \\ 0, & x_1, & 0, & x_2 \\ 0, & 0, & x_1, & 0 \\ 0, & 0, & 0, & x_1 \end{vmatrix}(\xi, \xi, \xi, \xi),$$

constituted by the aggregate of transformations $X=x_1A_1+x_2A_2$, is not similar to the group

$$(\xi_1', \xi_2', \xi_3', \xi_4')=\begin{vmatrix} y_1, & 0, & 0, & y_2 \\ 0, & y_1, & 0, & 0 \\ 0, & 0, & y_1, & 0 \\ 0, & 0, & 0, & y_1 \end{vmatrix}(\xi_1, \xi_2, \xi_3, \xi_4),$$

constituted by the aggregate of transformations $Y=y_1B_1+y_2B_2$.

From what has been shown above we have now the following theorem:

THEOREM III. *If $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ are two systems of linearly independent linear homogeneous transformations in n variables constituting the same hypercomplex number system, so that*

$$A_iA_j=\Sigma_{k=1}^{m}\gamma_{ijk}A_k,\ B_iB_j=\Sigma_{k=1}^{m}\gamma_{ijk}B_k,\qquad (i, j=1, 2, \ldots, m),$$

and if, moreover,

$$\bar{S}A_i=\bar{S}B_i,\qquad (i=1, 2, \ldots, m),$$

then the groups G and H, constituted by the aggregates of transformations $\Sigma_{i=1}^{m}x_iA_i$ and $\Sigma_{i=1}^{m}y_iB_i$, respectively, for all values of the x's and the y's are similar, provided

$$\begin{vmatrix} \bar{S}A_1A_1, & \bar{S}A_1A_2, & \ldots, & \bar{S}A_1A_m \\ \bar{S}A_2A_1, & \bar{S}A_2A_2, & \ldots, & \bar{S}A_2A_m \\ \ldots & \ldots & \ldots & \ldots \\ \bar{S}A_mA_1, & \bar{S}A_mA_2, & \ldots, & \bar{S}A_mA_m \end{vmatrix}=\begin{vmatrix} \bar{S}B_1B_1, & \bar{S}B_1B_2, & \ldots, & \bar{S}B_1B_m \\ \bar{S}B_2B_1, & \bar{S}B_2B_2, & \ldots, & \bar{S}B_2B_m \\ \ldots & \ldots & \ldots & \ldots \\ \bar{S}B_mB_1, & \bar{S}B_mB_2, & \ldots, & \bar{S}B_mB_m \end{vmatrix}\neq 0,$$

but not, in general, otherwise.

§ 3. *Conditions Sufficient for Complete Reducibility with Respect to a Given Domain of Rationality.*

In this and the following sections of this paper, R will denote a domain of rationality determined by a finite or infinite number of scalars; *e. g.*, R may denote the domain of rational numbers, or the domain of all real numbers, or even the domain of all scalars real and imaginary. A linear homogeneous transformation,

$$\begin{aligned}(\xi_1', \xi_2', \ldots, \xi_n') &= \Sigma_{\mu=1}^{n} \Sigma_{\nu=1}^{n} c_{\mu\nu} T_{\mu\nu}(\xi_1, \xi_2, \ldots, \xi_n) \\ &= \left(\begin{array}{l} c_{11}, c_{12}, \ldots, c_{1n} \\ c_{21}, c_{22}, \ldots, c_{2n} \\ \ldots\ldots\ldots\ldots \\ c_{n1}, c_{n2}, \ldots, c_{nn} \end{array} \right\rvert \xi_1, \xi_2, \ldots, \xi_n),\end{aligned}$$

will be said *to belong to R* or to be *rational with respect to R* if its coefficients are rational with respect to R. I shall assume that the linearly independent linear homogeneous transformations $A_1, A_2, \ldots, A_m$ constituting a hypercomplex number system belong to R. Then the constants γ_{ijk} $(i, j, k = 1, 2, \ldots, m)$ of multiplication of the number system $(A_1, A_2, \ldots, A_m)$ are rational with respect to R.* A number $X = \Sigma_{i=1}^{m} x_i A_i$ of this system is a transformation belonging to R if, and only if, the x's are rational with respect to R.† It follows from (5) that the aggregate of numbers of the system $(A_1, A_2, \ldots, A_m)$ that are transformations belonging to R constitute a group. This group will be denoted in what follows by $\overline{G}$; and G will, as in § 1 and § 2, denote the aggregate, constituting a group, of all numbers $\Sigma_{i=1}^{m} x_i A_i$ whatever of this system. The group $\overline{G}$ is obviously a subgroup of G. By what was stated, p. 337, it follows that the sum of two transformations of $\overline{G}$ is also a transformation of $\overline{G}$.

The group $\overline{G}$ is said to be *reducible with respect to R* if a linear substitution $\overline{\omega}$, one and the same for every transformation $X = \Sigma_{i=1}^{m} x_i A_i$ of $\overline{G}$ (that is, with constant coefficients), can be found such that

$$X = \Sigma_{i=1}^{m} x_i A_i = \overline{\omega} \begin{vmatrix} X_{11}, & X_{12} \\ 0, & X_{22} \end{vmatrix} \overline{\omega}^{-1},$$

where X_{11}, X_{22} are square matrices of order p and $n-p$ respectively $(1 < p < n)$,

---

* For the coefficients of the linear equations (4) are rational with respect to R; and these equations determine the γ's uniquely, since, otherwise, the A's are not linearly independent.

† Since the A's belong to R, it follows, see equation (3), that $a_{\mu\nu}^{(i)}$ is rational with respect to R for $i = 1, 2, \ldots, m$ and $\mu, \nu = 1, 2, \ldots, n$. Therefore, $\Sigma_{i=1}^{m} x_i a_{\mu\nu}^{(i)}$, for $\mu, \nu = 1, 2, \ldots, n$, is rational with respect to R, and thus $\Sigma_{i=1}^{m} x_i A_i$ belongs to R, if the x's are rational with respect to R. Conversely, if $\Sigma_{i=1}^{m} x_i A_i$ belongs to R, in which case $\Sigma_{i=1}^{m} x_i a_{\mu\nu}^{(i)}$ is rational with respect to R for $\mu, \nu = 1, 2, \ldots, n$, then the x's are rational with respect to R. For the x's are then determined by n^2 linear equations whose coefficients are rational with respect to R; and these equations determine the x's uniquely, since, otherwise, the A's are not linearly independent.

X_{12} is a rectangular matrix with p rows and $n-p$ columns, 0 denotes a rectangular matrix with $n-p$ rows and p columns whose constituents are all zero, and the constituents of X_{11}, X_{12} and X_{22} are rational with respect to R. Otherwise, $\overline{G}$ is *irreducible with respect to R.* If $\overline{G}$ is reducible with respect to R, the constituents of $\overline{\omega}$ can be taken rational with respect to R, since the equations determining these constituents are linear with coefficients belonging to R.

The aggregate of transformations X_{11} and the aggregate of transformations X_{22}, for all transformations X of $\overline{G}$, constitute groups $\overline{G}^{(1)}$, $\overline{G}^{(2)}$, respectively, of linear homogeneous transformations belonging to R in p and $n-p$ variables, respectively. Either $\overline{G}^{(1)}$ or $\overline{G}^{(2)}$, or both, may be reducible with respect to R. Ultimately, if $\overline{G}$ is reducible with respect to R, a linear substitution ω with constant coefficients can be found (which can be so chosen that its coefficients shall lie in R) such that

$$X=\Sigma_{i=1}^{m} x_i A_i=\omega \begin{vmatrix} X_{11}, & X_{12}, & \ldots, & X_{1\kappa} \\ 0, & X_{22}, & \ldots, & X_{2\kappa} \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & X_{\kappa\kappa} \end{vmatrix} \omega^{-1} \tag{144}$$

for any transformation X of $\overline{G}$, where the zeros denote rectangular matrices whose constituents are all zero, $X_{11}, X_{22}, \ldots, X_{\kappa\kappa}$ are square matrices, $X_{12}, \ldots, X_{\kappa-1,\kappa}$ rectangular matrices, whose constituents are rational with respect to R, and the aggregate of transformations X_{uu} $(u=1, 2, \ldots, \varkappa)$ for all transformations X of $\overline{G}$, constituting a group $\overline{G}_u$ irreducible with respect to R.

The group $\overline{G}$, if reducible with respect to R, is said to be *completely reducible with respect to R*, if ω can be so chosen that

$$X_{uv}=0, \qquad (u=1, 2, \ldots, \varkappa-1;\ v=u+1, u+2, \ldots, \varkappa);$$

otherwise *not completely reducible with respect to R.* The case in which $\overline{G}$ is irreducible with respect to R may be regarded as comprised in the case of complete reducibility, namely, when $\varkappa=1$; and following Loewy, I shall regard complete reducibility as embracing irreducibility.*

Since the coefficients of the transformations $A_1, A_2, \ldots, A_m$ of $\overline{G}$ are rational with respect to R, it follows that the constituents of the determinant

$$\nabla \equiv \begin{vmatrix} \bar{S}A_1A_1, & \bar{S}A_1A_2, & \ldots, & \bar{S}A_1A_m \\ \bar{S}A_2A_1, & \bar{S}A_2A_2, & \ldots, & \bar{S}A_2A_m \\ \cdots & \cdots & \cdots & \cdots \\ \bar{S}A_mA_1, & \bar{S}A_mA_2, & \ldots, & \bar{S}A_mA_m \end{vmatrix} \tag{145}$$

are rational with respect to R; and, since the constants γ_{ijk} $(i, j, k=1, 2, \ldots, m)$

---

* *Trans. Am. Math. Soc.*, Vol. VI (1905), p. 506.

of multiplication of the number system $(A_1, A_2, \ldots., A_m)$ are rational with respect to R, the constituents $S_1 A_i A_j$ and $S_2 A_i A_j$ $(i, j=1, 2, \ldots., m)$ of the determinants Δ_1 and Δ_2 are rational with respect to R. I shall now show that the group $\overline{G}$ is completely reducible with respect to R if $\nabla \neq 0$, in which case $\Delta_1 = \Delta_2 \neq 0$.

For, let $\nabla \neq 0$, in which case the number system $(A_1, A_2, \ldots., A_m)$ contains no invariant nilpotent subsystem. Either $\overline{G}$ is irreducible with respect to R, in which case the theorem is true, or $\overline{G}$ is reducible with respect to R. Let it be assumed that $\overline{G}$ is reducible with respect to R; and let

$$A_i = \omega \begin{vmatrix} A_{11}^{(i)}, & A_{12}^{(i)}, & \ldots., & A_{1\kappa}^{(i)} \\ 0, & A_{22}^{(i)}, & \ldots., & A_{2\kappa}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots., & A_{\kappa\kappa}^{(i)} \end{vmatrix} \omega^{-1}, \qquad (i=1, 2, \ldots., m), \tag{146}$$

where ω and $A_{uv}^{(i)}$ $(u=1, 2, \ldots., \varkappa;\ v=u, u+1, \ldots., \varkappa)$ belong to R. Then

$$X_{uv} = \Sigma_{i=1}^{m} x_i A_{uv}, \qquad (u=1, 2, \ldots., \varkappa;\ v=u, u+1, \ldots., \varkappa), \tag{147}$$

as will be seen on comparing with (144).

Let the order of the matrices $A_{uu}^{(1)}, A_{uu}^{(2)}, \ldots., A_{uu}^{(m)}$ $(u=1, 2, \ldots., \varkappa)$ be n_u. Then

$$n = n_1 + n_2 + \ldots. + n_\kappa. \tag{148}$$

If now, we denote by $\bar{S}^{(u)} A_{uu}^{(i)}$ for $1 \leqq i \leqq m$ and $1 \leqq u \leqq \varkappa$ the sum of the constituents in the principal diagonal of $A_{uu}^{(i)}$ divided by n_u, then, by (25),

$$\bar{S} A_i = \bar{S} \omega^{-1} A_i \omega = \frac{1}{n} [n_1 \bar{S}^{(1)} A_{11}^{(i)} + n_2 \bar{S}^{(2)} A_{22}^{(i)} + \ldots. + n_\kappa \bar{S}^{(\kappa)} A_{\kappa\kappa}^{(\kappa)}], \qquad (i=1, 2, \ldots., m). \tag{149}$$

From (146) and (1), it follows that

$$\begin{vmatrix} A_{11}^{(i)} A_{11}^{(j)}, & \Sigma_{w=1}^{2} A_{1w}^{(i)} A_{w2}^{(j)}, & \ldots., & \Sigma_{w=1}^{\kappa} A_{1w}^{(i)} A_{w\kappa}^{(j)} \\ 0, & A_{22}^{(i)} A_{22}^{(j)}, & \ldots., & \Sigma_{w=2}^{\kappa} A_{2w}^{(i)} A_{w\kappa}^{(j)} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots., & A_{\kappa\kappa}^{(i)} A_{\kappa\kappa}^{(j)} \end{vmatrix}$$
$$= \omega^{-1} A_i \omega \cdot \omega^{-1} A_j \omega = \omega^{-1} A_i A_j \omega = \Sigma_{k=1}^{m} \gamma_{ijk} \omega^{-1} A_k \omega$$
$$= \begin{vmatrix} \Sigma_{k=1}^{m} \gamma_{ijk} A_{11}^{(k)}, & \Sigma_{k=1}^{m} \gamma_{ijk} A_{12}^{(k)}, & \ldots., & \Sigma_{k=1}^{m} \gamma_{ijk} A_{1\kappa}^{(k)} \\ 0, & \Sigma_{k=1}^{m} \gamma_{ijk} A_{22}^{(k)}, & \ldots., & \Sigma_{k=1}^{m} \gamma_{ijk} A_{2\kappa}^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots., & \Sigma_{k=1}^{m} \gamma_{ijk} A_{\kappa\kappa}^{(k)} \end{vmatrix} \qquad (i, j=1, 2, \ldots., m); \tag{150}$$

and, therefore,

$$\Sigma_{w=u}^{v} A_{uw}^{(i)} A_{wv}^{(j)} = \Sigma_{k=1}^{m} \gamma_{ijk} A_{uv}^{(k)}, \qquad (u=1, 2, \ldots., \varkappa;\ v=u, u+1, \ldots., \varkappa). \tag{151}$$

In particular,

$$A^{(i)}_{uu}\,A^{(j)}_{uu}=\Sigma^{m}_{k=1}\gamma_{ijk}\,A^{(k)}_{uu}, \qquad (u=1, 2, \ldots., \varkappa). \tag{152}$$

Let now

$$B_i=\omega\left|\begin{array}{cccc} A^{(i)}_{11}, & 0, & \ldots., & 0 \\ 0, & A^{(i)}_{22}, & \ldots., & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots., & A^{(i)}_{\varkappa\varkappa} \end{array}\right|\omega^{-1}, \qquad (i=1, 2, \ldots., m). \tag{153}$$

The linear homogeneous transformations $B_1, B_2, \ldots., B_m$ then belong to R; and, by (152),

$$B_iB_j=\omega\left|\begin{array}{cccc} A^{(i)}_{11}A^{(j)}_{11}, & 0, & \ldots., & 0 \\ 0, & A^{(i)}_{22}A^{(j)}_{22}, & \ldots., & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots., & A^{(i)}_{\varkappa\varkappa}A^{(j)}_{\varkappa\varkappa} \end{array}\right|\omega^{-1}$$

$$=\omega\left|\begin{array}{cccc} \Sigma^{m}_{i=1}\gamma_{ijk}A^{(k)}_{11}, & 0, & \ldots., & 0 \\ 0, & \Sigma^{m}_{i=1}\gamma_{ijk}A^{(k)}_{22}, & \ldots., & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots., & \Sigma^{m}_{i=1}\gamma_{ijk}A^{(k)}_{\varkappa\varkappa} \end{array}\right|\omega^{-1}$$

$$=\Sigma^{m}_{k=1}\gamma_{ijk}B_k, \qquad (i, j=1, 2, \ldots., m). \tag{154}$$

Further,

$$\bar{S}B_i=\bar{S}\,\omega^{-1}B_i\,\omega=\frac{1}{n}\,[n_1\bar{S}^{(1)}A^{(i)}_{11}+n_2\bar{S}^{(2)}A^{(i)}_{22}+\ldots.+n_\varkappa\bar{S}^{(\varkappa)}A^{(i)}_{\varkappa\varkappa}]=\bar{S}A_i,$$

$$(i=1, 2, \ldots., m)\,; \tag{155}$$

and, therefore,

$$\bar{S}B_iB_j=\Sigma^{m}_{i=1}\gamma_{ijk}\bar{S}B_k=\Sigma^{m}_{i=1}\gamma_{ijk}\bar{S}A_k=\bar{S}A_iA_j, \qquad (i=1, 2, \ldots., m). \tag{156}$$

Whence it follows that

$$\left|\begin{array}{ccc} \bar{S}B_1B_1, & \ldots., & \bar{S}B_1B_m \\ \cdots & \cdots & \cdots \\ \bar{S}B_mB_1, & \ldots., & \bar{S}B_mB_m \end{array}\right|=\left|\begin{array}{ccc} \bar{S}A_1A_1, & \ldots., & \bar{S}A_1A_m \\ \cdots & \cdots & \cdots \\ \bar{S}A_mA_1, & \ldots., & \bar{S}A_mA_m \end{array}\right|\neq 0. \tag{157}$$

Finally, the matrices $B_1, B_2, \ldots., B_m$ are linearly independent. For, otherwise

$$c_1B_1+c_2B_2+\ldots.+c_mB_m=0,$$

where the c's are not all zero, in which case,

$$(c_1B_1+c_2B_2+\ldots.+c_mB_m)\,B_i=0, \qquad (i=1, 2, \ldots., m).$$

Therefore,

$$\begin{aligned} &c_1\bar{S}A_1A_i+c_2\bar{S}A_2A_i+\ldots.+c_m\bar{S}A_mA_i \\ &=c_1\bar{S}B_1B_i+c_2\bar{S}B_2B_i+\ldots.+c_m\bar{S}B_mB_i \\ &=\bar{S}(c_1B_1+c_2B_2+\ldots.+c_mB_m)B_i=0, \qquad (i=1, 2, \ldots., m)\,; \end{aligned}$$

and thus $\nabla=0$, which is contrary to supposition.

We have now shown that the two systems of linearly independent linear homogeneous transformations $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ in n variables constitute number systems having the same constants of multiplication and containing no invariant nilpotent subsystems; and that $\bar{S} A_i = \bar{S} B_i$ for $i = 1, 2, \ldots, m$. Therefore, by *Theorem III*, a linear substitution ω' with constant coefficients can be found such that

$$\Sigma_{i=1}^{m} x_i A_i = \omega' \Sigma_{i=1}^{m} x_i B_i \omega'^{-1} \tag{158}$$

for every system of values of the m scalars $x_1, x_2, \ldots, x_m$. Whence, if

$$\Omega = \omega' \omega, \tag{159}$$

it follows that

$$\Sigma_{i=1}^{m} x_i A = \Omega \begin{vmatrix} \Sigma_{i=1}^{m} x_i A_{11}^{(i)}, & 0, & \ldots, & 0 \\ 0, & \Sigma_{i=1}^{m} x_i A_{22}^{(i)}, & \ldots, & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0, & 0, & \ldots, & \Sigma_{i=1}^{m} x_i A_{\kappa\kappa}^{(i)} \end{vmatrix} \Omega^{-1}. \tag{160}$$

A fortiori, this is true for all transformations $\Sigma_{i=1}^{m} x_i A_i$ of G for which the x's are rational with respect to R, that is for all transformations of the subgroup $\bar{G}$ of G. Whence it follows that $\bar{G}$ is completely reducible with respect to R. We have, therefore, the following theorem:

THEOREM IV. *If $A_1, A_2, \ldots, A_m$ are m linearly independent linear homogeneous transformations in n variables whose coefficients are rational with respect to an arbitrarily given domain of rationality R, and if $A_1, A_2, \ldots, A_m$ constitute a hypercomplex number system, so that*

$$A_i A_j = \Sigma_{i=1}^{m} \gamma_{ijk} A_k, \qquad (i, j = 1, 2, \ldots, m),$$

the aggregate of transformations $\Sigma_{i=1}^{m} x_i A_i$, for all values of the x's rational with respect to R, constitute a group which is completely reducible with respect to R provided

$$\begin{vmatrix} \bar{S} A_1 A_1, & \bar{S} A_1 A_2, & \ldots, & \bar{S} A_1 A_m \\ \bar{S} A_2 A_1, & \bar{S} A_2 A_2, & \ldots, & \bar{S} A_2 A_m \\ \multicolumn{4}{c}{\dotfill} \\ \bar{S} A_m A_1, & \bar{S}_m A_2 A, & \ldots, & \bar{S} A_m A_m \end{vmatrix} \neq 0,$$

or, what is the same thing, provided

$$\begin{vmatrix} S_1 A_1 A_1, & S_1 A_1 A_2, & \ldots, & S_1 A_1 A_m \\ S_1 A_2 A_1, & S_1 A_2 A_2, & \ldots, & S_1 A_2 A_m \\ \multicolumn{4}{c}{\dotfill} \\ S_1 A_m A_1, & S_1 A_m A_2, & \ldots, & S_1 A_m A_m \end{vmatrix} = \begin{vmatrix} S_2 A_1 A_1, & S_2 A_1 A_2, & \ldots, & S_2 A_1 A_m \\ S_2 A_2 A_1, & S_2 A_2 A_2, & \ldots, & S_2 A_2 A_m \\ \multicolumn{4}{c}{\dotfill} \\ S_2 A_m A_1, & S_2 A_m A_2, & \ldots, & S_2 A_m A_m \end{vmatrix} \neq 0,$$

§ 4. *Conditions Necessary and Sufficient for Complete Reducibility with Respect to a Given Domain of Rationality.*

Employing the notation of the preceding section, let the m linearly independent linear homogeneous transformations $A_1, A_2, \ldots, A_m$, constituting a hypercomplex number system, belong to R. I shall now assume that $\nabla = 0$; and I shall show that then $\overline{G}$ is reducible, but not completely reducible with respect to R.

Let the nullity of ∇ be m_0 $(1 \leqq m_0 \leqq m)$. There is then a system of just m_0 linearly independent numbers,

$$N_u = \alpha_{u1} A_1 + \alpha_{u2} A_2 + \ldots + \alpha_{um} A_m, \qquad (u = 1, 2, \ldots, m_0), \tag{161}$$

of the number system $(A_1, A_2, \ldots, A_m)$ satisfying the equations

$$\begin{aligned} &\overline{S}(x_1 A_1 + x_2 A_2 + \ldots + x_m A_m) A_i \\ &= x_1 \overline{S} A_1 A_i + x_2 \overline{S} A_2 A_i + \ldots + x_m \overline{S} A_m A_i = 0, \qquad (i = 1, 2, \ldots, m), \end{aligned} \tag{162}$$

and constituting a maximal invariant nilpotent subsystem of $(A_1, A_2, \ldots, A_m)$.* Since, as stated above, p. 359, the coefficients, $S A_i A_j$ $(i, j = 1, 2, \ldots, m)$, of the linear equations determining the N's are rational with respect to R, we may so choose $N_1, N_2, \ldots, N_{m_0}$ that they shall belong to R. Let the N's be so chosen. Since they constitute an invariant subsystem of $(A_1, A_2, \ldots, A_m)$ it follows that

$$N_u N_v = \Sigma_{w=1}^{m_0} \gamma_{uvw}^{(0)} N_w, \qquad (u, v = 1, 2, \ldots, m_0), \tag{163}$$

$$A_i N_u = \Sigma_{v=1}^{m_0} \gamma_{iuv}^{(1)} N_v \qquad (i = 1, 2, \ldots, m;\ u = 1, 2, \ldots, m_0), \tag{164}$$

where, since $A_1, A_2, \ldots, A_m$ and $N_1, N_2, \ldots, N_{m_0}$ belong to R, the γ's are rational with respect to R.

The linear homogeneous transformations $1, N_1, N_2, \ldots, N_{m_0}$ are linearly independent† and by (163) constitute a number system (of which 1 is the modulus) whose constants of multiplication are rational with respect to R; and thus, since $N_1, N_2, \ldots, N_{m_0}$ belong to R, the aggregate of transformations

$$Y = y_0 + \Sigma_{u=1}^{m_0} y_u N_u, \tag{165}$$

for any systems of scalars $y_0, y_1, \ldots, y_{m_0}$ rational with respect to R, constitute a group $\overline{H}$ of transformations belonging to R. Since $N_1, N_2, \ldots, N_{m_0}$

---

* See p. 343.

† For if

$$c_0 1 + \Sigma_i^{m_0} c_i N_i = 0,$$

then

$$c_0 = c_0 \overline{S} 1 + \Sigma_{i=1}^{m_0} c_i \overline{S} N_i = \overline{S}(c_0 1 + \Sigma_{i=1}^{m_0} c_i N_i) = 0.$$

Therefore, $\Sigma_{i=1}^{m_0} c_i N_i = 0$; and thus

$$c_1 = c_2 = \ldots\ c_{m_0} = 0.$$

are transformations of $\overline{G}$, the group $\overline{H}$ is a subgroup of $\overline{G}$. From *Theorem VI*, proved in the next section, it follows that a linear function,

$$N=c_1N_1+c_2N_2+\dots\dots c_{m_0}N_{m_0}\neq 0, \tag{166}$$

of the N's with coefficients rational with respect to R, can be found such that

$$N_uN=0, \qquad (u=1,2,\dots\dots,m_0)\,; \tag{167}$$

and that a linear substitution ω can be found whose coefficients are rational with respect to R such that

$$N_u=\omega\begin{vmatrix}0, & b_{12}^{(u)}, & \dots\dots, & b_{1n}^{(u)}\\ 0, & 0, & \dots\dots, & b_{2n}^{(u)}\\ \dots & \dots & \dots & \dots\\ 0, & 0, & \dots\dots, & 0\end{vmatrix}\omega^{-1}, \qquad (u=1,2,\dots\dots,m_0), \tag{168}$$

where the b's are all rational with respect to R, in which case

$$N=\Sigma_{u=1}^{m_0}c_uN_u=\omega\begin{vmatrix}0, & b_{12}, & \dots\dots, & b_{1n}\\ 0, & 0, & \dots\dots, & b_{2n}\\ \dots & \dots & \dots & \dots\\ 0, & 0, & \dots\dots, & 0\end{vmatrix}\omega^{-1}, \tag{169}$$

where $b_{\mu\nu}$ $(\mu=1,2,\dots\dots,n-1\,;\ \nu=\mu+1,\mu+2,\dots\dots,n)$ is rational with respect to R. By the aid of this theorem, we may now show that $\overline{G}$ is reducible with respect to R.

Thus let $X=\Sigma_{i=1}^{m}x_iA_i$, where $x_1,x_2,\dots\dots,x_m$ are rational with respect to R, be any transformation of $\overline{G}$; and let

$$\omega^{-1}X\omega=\begin{vmatrix}\beta_{11}(x), & \beta_{12}(x), & \dots\dots, & \beta_{1n}(x)\\ \beta_{21}(x), & \beta_{22}(x), & \dots\dots, & \beta_{2n}(x)\\ \dots & \dots & \dots & \dots\\ \beta_{n1}(x), & \beta_{n2}(x), & \dots\dots, & \beta_{nn}(x)\end{vmatrix}. \tag{170}$$

Then $\beta_{11}(x),\ \beta_{12}(x),\dots\dots,\ \beta_{nn}(x)$ are rational with respect to R.

By equation (164),

$$\begin{aligned}XN&=\Sigma_{i=1}^{m}x_iA_i\cdot\Sigma_{u=1}^{m_0}c_uN_u\\ &=\Sigma_{i=1}^{m}\Sigma_{u=1}^{m_0}x_ic_uA_iN_u\\ &=\Sigma_{i=1}^{m}\Sigma_{u=1}^{m_0}\Sigma_{v=1}^{m_0}x_ic_u\gamma_{iuv}N_v=\Sigma_{v=1}^{m_0}x_v'N_v,\end{aligned} \tag{171}$$

provided

$$x_v'=\Sigma_{i=1}^{m}\Sigma_{u=1}^{m_0}x_ic_u\gamma_{iuv}, \qquad (v=1,2,\dots\dots,m_0)\,; \tag{172}$$

and, therefore,

$$\begin{aligned}\omega^{-1}X\omega\cdot\omega^{-1}N\omega&=\omega^{-1}XN\omega\\ &=\omega^{-1}\Sigma_{v=1}^{m_0}x_v'N_v\omega\\ &=\Sigma_{v=1}^{m_0}x_v'\omega^{-1}N_v\omega.\end{aligned} \tag{173}$$

Whence it follows that every constituent of $\omega^{-1}X\omega\cdot\omega^{-1}N\omega$ in and below the principal diagonal is equal to zero, since this is true for $\Sigma_{v=1}^{m_0} x'_v \omega^{-1} N_v \omega$. That is,

$$\beta_{\mu 1}(x)\, b_{1\nu} + \ldots . + \beta_{\mu\lambda}(x)\, b_{\lambda\nu} + \ldots . + \beta_{\mu\nu-1}(x)\, b_{\nu-1,\nu} = 0,$$
$$(\mu = 2, 3, \ldots ., n;\ \nu = 2, 3, \ldots ., \mu). \qquad (174)$$

Not every constituent of $\omega^{-1} N \omega$ is zero, since otherwise $N = 0$. We may, therefore, assume that $b_{\lambda\nu'} \neq 0$ for $1 < \nu' \leqq n$ and $\lambda < \nu'$.

Therefore, not every coefficient is zero in the linear relation we obtain from (174) by putting $\nu = \nu'$. That is to say, the same linear relation, rational with respect to R, obtains between the first $\nu' - 1$ constituents of the μ-th row ($\mu \geqq \nu'$) of every transformation of the group of linear homogeneous transformations $\omega^{-1} X \omega$ belonging to R. But then by Maschke's theorem, this group is reducible with respect to R.* Whence it follows that $\overline{G}$ is reducible with respect to R. That is to say, *if $A_1, A_2, \ldots ., A_m$ are m linearly independent linear homogeneous transformations which belong to R and constitute a hypercomplex number system containing an invariant nilpotent subsystem, the group of transformations belonging to R constituted by the aggregate of transformations $\Sigma_{i=1}^{m} x_i A_i$, for all values of the x's rational with respect to R, is reducible with respect to R.*

If possible, let $\overline{G}$ be completely reducible with respect to R. Then, employing the notation of § 3, for any transformation of $\overline{G}$ (that is, for $x_1, x_2, \ldots ., x_m$ rational with respect to R), we have

$$X = \Sigma_{i=1}^{m} x_i A_i = \omega \begin{vmatrix} \Sigma_{i=1}^{m} x_i A_{11}^{(i)}, & 0, & \ldots ., & 0 \\ 0, & \Sigma_{i=1}^{m} x_i A_{22}^{(i)}, & \ldots ., & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots ., & \Sigma_{i=1}^{m} x_i A_{\kappa\kappa}^{(i)} \end{vmatrix} \omega^{-1}, \qquad (175)$$

where ω and the matrices $A_{uu}^{(i)}$ for $i = 1, 2, \ldots ., m$ and $u = 1, 2, \ldots ., \kappa$ belong to R, and the group $\overline{G}_u$ $(1 \leqq u \leqq \kappa)$ of linear homogeneous transformations $\Sigma_{i=1}^{m} x_i A_{uu}^{(i)}$ for values of the x's rational with respect to R is irreducible with respect to R.

---

* *Math. Ann.*, Vol. LII (1889). In this paper Maschke does not refer explicitly to a domain of rationality. But, in the proof of the theorem given therein, p. 366, he shows that, if a non-diagonal constituent is zero throughout in the matrix of a group of linear homogeneous transformations, the group is reducible with respect to the domain of rationality in which the coefficients lie.

Let now $X^{(1)}$ denote the matrix or linear homogeneous transformation $\omega^{-1} X \omega$ whose constituent in the u-th row and v-th column is $\beta_{uv}(x)$ $(u, v = 1, 2, \ldots ., n)$; and let $\overline{\omega}$ denote the linear substitution defined by the equations

$$\xi'_i = \xi_i + b_{i\nu'} \xi_\lambda, \quad \xi'_\lambda = b_{\lambda\nu'} \xi_\lambda, \quad \xi'_{\nu'+j} = \xi_{\nu'+j},$$
$$(i = 1, 2, \ldots ., \lambda - 1, \lambda + 1, \ldots ., \nu' - 1;\ j = 0, 1, \ldots ., n - \nu').$$

Then $\overline{\omega}$ belongs to R; and each of the group of transformations $\overline{\omega}^{-1} X^{(1)} \overline{\omega}$ (for all transformations X of G) belongs to R. Let $\mu \geqq \nu'$. Then, since $\Sigma_{i=1}^{\nu'-1} b_{i\nu'} \beta_{\mu i}(x) = 0$, it follows that the constituent in the μ-th row and λ-th column of the matrix of $\overline{\omega}^{-1} X^{(1)} \overline{\omega}$ is zero throughout. Therefore, by Maschke's theorem, this group is reducible with respect to R; and thus the group of transformations $X^{(1)} = \omega^{-1} X \omega$ is reducible with respect to R.

Let $1 \leqq u \leqq \kappa$; and let $m_u (1 \leqq m_u \leqq m)$ be the number of the linear homogeneous transformations $A_{uu}^{(1)}, A_{uu}^{(2)}, \ldots, A_{uu}^{(m)}$ that are linearly independent. Without loss of generality, we may assume that $A_{uu}^{(1)}, A_{uu}^{(2)}, \ldots, A_{uu}^{(m_u)}$ are linearly independent; and then

$$A_{uu}^{(m_u+h)} = c_{h1}^{(u)} A_{uu}^{(1)} + c_{h1}^{(u)} A_{uu}^{(2)} + \ldots + c_{hm_u}^{(u)} A_{uu}^{(m_u)}, \quad (h=1, 2, \ldots, m-m_u). \tag{176}$$

where the c's belong to R. Wherefore, by (152),

$$\begin{aligned} A_{uu}^{(i)} A_{uu}^{(j)} &= \Sigma_{k=1}^{m} \gamma_{ijk} A_{uu}^{(k)} \\ &= \Sigma_{k=1}^{m_u} \gamma_{ijk} A_{uu}^{(k)} + \Sigma_{h=1}^{m-m_u} \gamma_{i,j,m_u+h} \Sigma_{k=1}^{m_u} c_{hk}^{(u)} A_{uu}^{(k)} \\ &= \Sigma_{k=1}^{m_u} \bar{\gamma}_{ijk}^{(u)} A_{uu}^{(k)}, \qquad (i, j=1, 2, \ldots, m_u), \end{aligned} \tag{177}$$

provided

$$\bar{\gamma}_{ijk}^{(u)} = \gamma_{ijk} + \Sigma_{h=1}^{m-m_u} \gamma_{i,j,m_u+h} c_{hk}^{(u)}, \qquad (i, j, k=1, 2, \ldots, m_u). \tag{178}$$

Whence it follows that the linear homogeneous transformations $A_{uu}^{(1)}, A_{uu}^{(2)}, \ldots, A_{uu}^{(m_u)}$ belonging to R constitute a number system whose constants $\bar{\gamma}_{ijk}^{(u)}$ $(i, j, k= 1, 2, \ldots, m_u)$ of multiplication are rational with respect to R. From (176) it follows that the group $\bar{G}_u$ is constituted by the aggregate of numbers $\Sigma_{i=1}^{m_u} x_i A_{uu}^{(i)}$ of this number system for values of the x's rational with respect to R. For this number system the determinant

$$\nabla^{(u)} \equiv \begin{vmatrix} \bar{S}^{(u)} A_{uu}^{(1)} A_{uu}^{(1)}, & \ldots, & \bar{S}^{(u)} A_{uu}^{(1)} A_{uu}^{(m_u)} \\ \ldots & \ldots & \ldots \\ \bar{S}^{(u)} A_{uu}^{(m_u)} A_{uu}^{(1)}, & \ldots, & \bar{S}^{(u)} A_{uu}^{(m_u)} A_{uu}^{(m_u)} \end{vmatrix} \neq 0; \tag{179}$$

that is, this number system contains no invariant nilpotent subsystem. For otherwise, by the theorem just proved, it follows that G_u is reducible with respect to R. Therefore, there is no number $\Sigma_{i=1}^{m_0} b_i A_{uu}^{(i)}$ of this number system whose product with every number of this system is nilpotent. *A fortiori*, there is no transformation $\Sigma_{i=1}^{m} b_i A_{uu}^{(i)}$ of $\bar{G}_u$ whose product with $\Sigma_{i=1}^{m} x_i A_{uu}^{(i)}$ is nilpotent for every system of values of the x's.

Let now

$$N = \Sigma_{u=1}^{m_0} c_u N_u = \Sigma_{u=1}^{m_0} \Sigma_{i=1}^{m} c_u \alpha_{ui} A_i = \Sigma_{i=1}^{m} b_i A_i. \tag{180}$$

Then, by (175),

$$N = \omega \begin{vmatrix} \Sigma_{i=1}^{m} b_i A_{11}^{(i)}, & 0, & \ldots, & 0 \\ 0, & \Sigma_{i=1}^{m} b_i A_{22}^{(i)}, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & \Sigma_{i=1}^{m} b_i A_{\kappa\kappa}^{(i)} \end{vmatrix} \omega^{-1}. \tag{181}$$

Since, by (163),

$$\begin{aligned} XN &= \Sigma_{i=1}^{m} x_i A_i \cdot \Sigma_{u=1}^{m_0} c_u N_u \\ &= \Sigma_{i=1}^{m} \Sigma_{u=1}^{m_0} x_i c_u A_i N_u \\ &= \Sigma_{i=1}^{m} \Sigma_{u=1}^{m_0} \Sigma_{v=1}^{m_0} x_i c_u \gamma_{iuv}^{(1)} N_v \end{aligned} \tag{182}$$

is expressible linearly in terms of the units of the nilpotent subsystem $(N_1, N_2, \dots\dots, N_{m_0})$, it is nilpotent; and thus $\omega^{-1}X\omega\cdot\omega^{-1}N\omega$ is nilpotent for any system of scalars $x_1, x_2, \dots\dots, x_m$. Whence, for each integer u from 1 to $\varkappa$, it follows that some power of the product of $\Sigma_{i=1}^{m} x_i A_{uu}^{(i)}$ and $\Sigma_{i=1}^{m} b_i A_{uu}^{(i)}$ is zero for any system of scalars $x_1, x_2, \dots\dots, x_m$; and thus, either the product of $\Sigma_{i=1}^{m} x_i A_{uu}^{(i)}$ and $\Sigma_{i=1}^{m} b_i A_{uu}^{(i)}$ is nilpotent for every system of scalars $x_1, x_2, \dots\dots, x_m$, which as shown above is impossible, or $\Sigma_{i=1}^{m} b_i A_{uu}^{(i)}=0$. But it can not happen that $\Sigma_{i=1}^{m} b_i A_{uu}^{(i)}=0$ for $u=1, 2, \dots\dots, \varkappa$; for then, by (181), $N=0$, which is contrary to supposition. Wherefore, the hypothesis is absurd, that $\overline{G}$ is completely reducible with respect to R.

From the theorem on p. 362 and what has just been proved, we obtain the following theorem:

THEOREM V. *If $A_1, A_2, \dots\dots, A_m$ are m linearly independent linear homogeneous transformations in n variables that belong to an arbitrarily given domain of rationality R and constitute a hypercomplex number system, so that*

$$A_i A_j=\Sigma_{k=1}^{m} \gamma_{ijk} A_k, \qquad (i, j=1, 2, \dots\dots, m),$$

the group $\overline{G}$ of transformations

$$X=\Sigma_{i=1}^{m} x_i A_i$$

for all values of the x's rational with respect to R, is completely reducible with respect to R if, and only if,

$$\begin{vmatrix} SA_1A_1, & SA_1A_2, & \dots\dots, & SA_1A_m \\ SA_2A_1, & SA_2A_2, & \dots\dots, & SA_2A_m \\ \dots & \dots & \dots & \dots \\ SA_mA_1, & SA_mA_2, & \dots\dots, & SA_mA_m \end{vmatrix} \neq 0;$$

and, therefore, if and only if,

$$\begin{vmatrix} S_1A_1A_1, & S_1A_1A_2, & \dots\dots, & S_1A_1A_m \\ S_1A_2A_1, & S_1A_2A_2, & \dots\dots, & S_1A_2A_m \\ \dots & \dots & \dots & \dots \\ S_1A_mA_1, & S_1A_mA_2, & \dots\dots, & S_1A_mA_m \end{vmatrix} = \begin{vmatrix} S_2A_1A_1, & S_2A_1A_2, & \dots\dots, & S_2A_1A_m \\ S_2A_2A_1, & S_2A_2A_2, & \dots\dots, & S_2A_2A_m \\ \dots & \dots & \dots & \dots \\ S_2A_mA_1, & S_2A_mA_2, & \dots\dots, & S_2A_mA_m \end{vmatrix} \neq 0.$$

§ 5.

As above, let the $m(m>1)$ linearly independent matrices $A_1, A_2, \dots\dots, A_m$ constitute a hypercomplex number system, so that

$$A_i A_j=\Sigma_{k=1}^{m} \gamma_{ijk} A_k, \qquad (i=1, 2, \dots\dots, m); \tag{183}$$

and, as in § 4, let $A_1, A_2, \dots\dots, A_m$ belong to R. Then the constants γ_{ijk} $(i, j, k=1, 2, \dots\dots, m)$ of multiplication of the number system $(A_1, A_2, \dots\dots, A_m)$ belong to R. I shall now assume that this number system contains but one idempotent number, the modulus 1.

In the *Transactions of the American Mathematical Society*, Vol. XXXIV (1904), pp. 529, 540, and 547, I have shown that in any number system, as $(A_1, A_2, \ldots, A_m)$, that contains a modulus but no other idempotent number, we may introduce new units—namely, the modulus,

$$1 = a_1^{(0)} A_1 + a_2^{(0)} A_2 + \ldots + a_m^{(0)} A_m, \tag{184}$$

and $m-1$ nilpotent numbers,

$$N_i = \tau_{1i} A_1 + \tau_{2i} A_2 + \ldots + \tau_{mi} A_m, \qquad (i = 1, 2, \ldots, m-1), \tag{185}$$

constituting an invariant nilpotent subsystem by a transformation rational with respect to the domain of rationality of the constants of multiplication of the system; and that there is a linear function,

$$N = c_1 N_1 + c_2 N_2 + \ldots + c_{m-1} N_{m-1} \neq 0, \tag{186}$$

with coefficients rational with respect to this domain, such that

$$N_i N = 0 = N N_i, \qquad (i = 1, 2, \ldots, m). \tag{187}$$

For such a transformation of the number system $(A_1, A_2, \ldots, A_m)$, the coefficients $a_i^{(0)}$, τ_{ij}, c_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m-1$, being rational with respect to the domain of rationality of the constants of multiplication of this system, are rational with respect to R. And, therefore, since the A's belong to R, it follows that N and the new units $1, N_1, N_2, \ldots, N_{m-1}$ belong to R. Whence it follows that the group $\overline{G}$ of transformations belonging to R that are numbers of the system $(A_1, A_2, \ldots, A_m)$ is constituted by the aggregate of transformations

$$Y = y_0 + \Sigma_{i=1}^{m-1} y_i N_i \tag{188}$$

for $y_0, y_1, \ldots, y_{m-1}$ rational with respect to R. Let

$$N_i N_j = \Sigma_{k=1}^{m-1} \gamma_{ijk}^{(0)} N_k, \qquad (i, j = 1, 2, \ldots, m-1). \tag{189}$$

Then γ_{ijk}, for $i, j, k = 1, 2, \ldots, m-1$, is rational with respect to R.

Since N belongs to R, and since N is nilpotent, and thus every root of its characteristic equation is zero, a linear substitution $\overline{\omega}$ belonging to R can be found such that

$$N = \overline{\omega} \begin{vmatrix} 0, & b_{12}, & \ldots, & b_{1n} \\ 0, & 0, & \ldots, & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & 0 \end{vmatrix} \overline{\omega}^{-1}, \tag{190}$$

where the b's are rational with respect to R. Let

$$Y = y_0 + \Sigma_{i=1}^{m-1} y_i N_i, \tag{191}$$

where the y's are rational with respect to R, be any transformation of $\overline{G}$; and let

$$\overline{\omega}^{-1}Y\overline{\omega}=y_0+\Sigma_{i=1}^{m-1}y_i\overline{\omega}^{-1}N_i\overline{\omega}=\begin{vmatrix}\beta_{11}(y), & \beta_{12}(y), & \ldots\ldots, & \beta_{1n}(y)\\ \beta_{21}(y), & \beta_{22}(y), & \ldots\ldots, & \beta_{2n}(y)\\ \ldots & \ldots & \ldots & \ldots\\ \beta_{n1}(y), & \beta_{n2}(y), & \ldots\ldots, & \beta_{nn}(y)\end{vmatrix}. \tag{192}$$

Then $\beta_{\mu\nu}(y)$, for $\mu, \nu=1, 2, \ldots\ldots, n$, is rational with respect to R. Since, by (187),

$$\begin{aligned}\overline{\omega}^{-1}Y\overline{\omega}\cdot\overline{\omega}^{-1}N\omega&=\overline{\omega}^{-1}YN\overline{\omega}\\ &=\overline{\omega}^{-1}(y_0+\Sigma_{i=1}^{m-1}y_iN_i)N\overline{\omega}\\ &=y_0\overline{\omega}^{-1}N\overline{\omega},\end{aligned} \tag{193}$$

it follows that the constituent in the μ-th row and ν-th column of $\overline{\omega}^{-1}Y\overline{\omega}\cdot\overline{\omega}^{-1}N\overline{\omega}$ is zero, provided $\nu\leqq\mu$; for this is true of $\overline{\omega}^{-1}N\overline{\omega}$. Therefore,

$$\beta_{\mu1}(y)\,b_{1\nu}+\ldots\ldots+\beta_{\mu\lambda}(y)\,b_{\lambda\nu}+\ldots\ldots+\beta_{\mu,\,\nu-1}(y)\,b_{\nu-1,\,\nu}=0,$$
$$(\mu=2, 3, \ldots\ldots, n;\ \nu=2, 3, \ldots\ldots, \mu). \tag{194}$$

Not every constituent of $\overline{\omega}^{-1}N\overline{\omega}$ is zero, since otherwise $N=0$. We may, therefore, then assume that $b_{\lambda\nu'}\neq0$ for $1<\nu'\leqq n$ and $\lambda<\nu'$. Therefore, not every constituent is zero in the linear relation we derive from (194) by putting $\nu=\nu'$; and thus the same rational relation with respect to R obtains between the first $\nu'-1$ constituents of the μ-th row ($\mu\geqq\nu'$) of every transformation of the group of linear homogeneous transformations $\overline{\omega}^{-1}Y\overline{\omega}$ belonging to R. But then, by Maschke's theorem, this group is reducible with respect to R.* Wherefore, $\overline{G}$ is reducible with respect to R. That is to say, *if $A_1, A_2, \ldots\ldots, A_m$ are m ($m>1$) linearly independent transformations which belong to R and constitute a hypercomplex number system containing a modulus but no other idempotent number, the group constituted by the aggregate of transformations $\Sigma_{i=1}^{m}x_iA_i$ for all values of the x's rational with respect to R, is reducible with respect to R.*

We have now shown that it is possible to find a linear substitution ω belonging to R such that, for any transformation $Y=y_0+\Sigma_{i=1}^{m-1}y_iN_i$ of $\overline{G}$, that is, for $y_0, y_1, \ldots\ldots, y_{m-1}$ any system of scalars rational with respect to R,

$$Y=y_0+\Sigma_{i=1}^{m-1}y_iN_i=\omega\begin{vmatrix}Y_{11}, & Y_{12}, & \ldots\ldots, & Y_{1\kappa}\\ 0, & Y_{22}, & \ldots\ldots, & Y_{2\kappa}\\ \ldots & \ldots & \ldots & \ldots\\ 0, & 0, & \ldots\ldots, & Y_{\kappa\kappa}\end{vmatrix}\omega^{-1}, \tag{195}$$

where Y_{uv} for $u=1, 2, \ldots\ldots, \varkappa$ and $v=u, u+1, \ldots\ldots, \varkappa$ are matrices whose constituents lie in R, and, for $1 \leqq u \leqq \varkappa$, the aggregate of transformations Y_{uu} belonging to R constitute a group $\overline{G}_u$ irreducible with respect to R.

---

* See note, p. 365.

Let

$$N_i = \omega \begin{vmatrix} N_{11}^{(i)}, & N_{12}^{(i)}, & \ldots, & N_{1\kappa}^{(i)} \\ 0, & N_{22}^{(i)}, & \ldots, & N_{2\kappa}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & N_{\kappa\kappa}^{(i)} \end{vmatrix} \omega^{-1}, \qquad (i=1, 2, \ldots, m-1), \tag{196}$$

where $N_{uv}^{(i)}$ $(u=1, 2, \ldots, \varkappa;\ v=u, u+1, \ldots, \varkappa)$ is rational with respect to R; and, for $1 \leqq u \leqq \varkappa$, let 1_u denote the identical transformation of the group $\overline{G}_u$ of transformations Y_{uu}. Then

$$Y_{uu} = y_0 1_u + \Sigma_{i=1}^{m-1} y_i N_{uu}^{(i)}, \qquad (u=1, 2, \ldots, \varkappa), \tag{197}$$

$$Y_{uv} = \Sigma_{i=1}^{m-1} y_i N_{uv}^{(i)}, \qquad (u=1, 2, \ldots, \varkappa;\ v=u+1, u+2, \ldots, \varkappa). \tag{198}$$

From (189) it follows that

$$\omega^{-1} N_i \omega \cdot \omega^{-1} N_j \omega = \omega^{-1} N_i N_j \omega = \Sigma_{k=1}^{m-1} \gamma_{ijk}^{(0)} \omega^{-1} N_k \omega,$$
$$(i, j=1, 2, \ldots, m-1); \tag{199}$$

and, therefore,

$$N_{uu}^{(i)} N_{uu}^{(j)} = \Sigma_{k=1}^{m-1} \gamma_{ijk}^{(0)} N_{uu}^{(k)}, \qquad (i, j=1, 2, \ldots, m-1;\ u=1, 2, \ldots, \varkappa). \tag{200}$$

Let $1 \leqq u \leqq \varkappa$; and let m_u $(0 \leqq m_u \leqq m-1)$ be the number of linear homogeneous transformations $N_{uu}^{(1)}, N_{uu}^{(2)}, \ldots, N_{uu}^{(m-1)}$ that are linearly independent. We shall now show that $m_u=0$. For, if possible let $m_u>0$, in which case we may assume with loss of generality that $N_{uu}^{(1)}, N_{uu}^{(2)}, \ldots, N_{uu}^{(m_u)}$ are linearly independent; and thus

$$N_{uu}^{(m_u+h)} = c_{h1}^{(u)} N_{uu}^{(1)} + c_{h2}^{(u)} N_{uu}^{(2)} + \ldots + c_{hm_u}^{(u)} N_{uu}^{(m_u)}, \qquad (h=1, 2, \ldots, m-m_u), \tag{201}$$

where the c's are rational with respect to R. From (197) it then follows that the group $\overline{G}_u$ of transformations belonging to R is constituted by the aggregate of numbers $y_0 + \Sigma_{i=1}^{m_u} y_i N_{uu}^{(i)}$ for values of the y's rational with respect to R; and, from (200), that $N_{uu}^{(1)}, N_{uu}^{(2)}, \ldots, N_{uu}^{(m_u)}$ constitute a hypercomplex number system.* This system is nilpotent. For $\Sigma_{i=1}^{m_u} y_i N_i$ and, therefore, $\Sigma_{i=1}^{m_u} y_i \omega^{-1} N_i \omega$, is nilpotent, for every system of values of the y's. Whence, by (196), it follows that $\Sigma_{i=1}^{m_u} y_i N_{uu}^{(i)}$ is nilpotent for every system of values of the y's. The linear homogeneous transformations $1_u, N_{uu}^{(1)}, N_{uu}^{(2)}, \ldots, N_{uu}^{(m_u)}$ are linearly independent and constitute a number system with but one idempotent number, the modulus 1_u.† Therefore, by the theorem proved p. 369, the group $\overline{G}_u$ of transformations belonging to R constituted by the aggregate of transformations $y_0 + \Sigma_{i=1}^{m_u} y_i N_{uu}^{(i)}$, for values of the y's rational with respect to R, is reducible with respect to R, which is contrary to supposition. Therefore, $m_u = 0$, and thus

---

* Cf. p. 366.

† See note, p. 363; also p. 372.

$$N_{uu}^{(i)} = 0, \qquad (i=1, 2, \ldots, m-1); \tag{202}$$

whence follows

$$Y_{uu} = y_0 1_u, \qquad (u=1, 2, \ldots, \varkappa). \tag{203}$$

The order of the matrix 1_u is equal to unity; otherwise, $\overline{G}_u$ is reducible with respect to R. Since this is true for each value of u from 1 to $\varkappa$, it follows that each of the matrices Y_{uv} $(u = 1, 2, \ldots, \varkappa;\ v = u, u+1, \ldots, \varkappa)$ is of order one. Whence it follows that $\varkappa = n$; and, by (203),

$$Y_{uu} = y_0, \qquad (u = 1, 2, \ldots, n). \tag{204}$$

Again, by (196) and (202),

$$N_i = \omega \begin{vmatrix} 0, & b_{12}^{(i)}, & \ldots, & b_{1n}^{(i)} \\ 0, & 0, & \ldots, & b_{2n}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & 0 \end{vmatrix} \omega^{-1}, \qquad (i=1, 2, \ldots, m-1), \tag{205}$$

where the b's are rational with respect to R. Wherefore, by (195),

$$Y = y_0 + \Sigma_{i=1}^{m-1} y_i N_i = \omega \begin{vmatrix} y_0, & \Sigma_{i=1}^{m-1} y_i b_{12}^{(i)}, & \ldots, & \Sigma_{i=1}^{m-1} y_i b_{1n}^{(i)} \\ 0, & y_0, & \ldots, & \Sigma_{i=1}^{m-1} y_i b_{2n}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & y_0 \end{vmatrix} \omega^{-1}. \tag{206}$$

The group $\overline{G}$ can not be completely reducible with respect to R. For it would then be possible to choose ω so that

$$Y_{uv} = 0, \qquad (u=1, 2, \ldots, \varkappa;\ v=u+1, u+2, \ldots, \varkappa);$$

and we should then have

$$Y = y_0 + \Sigma_{i=1}^{m-1} y_i N_i = \omega \begin{vmatrix} Y_{11}, & 0, & \ldots, & 0 \\ 0, & Y_{22}, & \ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & Y_{\kappa\kappa} \end{vmatrix} \omega^{-1}.$$

But it has been shown that $Y_{uu} = y_0 1_u$ for $u=1, 2, \ldots, \varkappa$; and we should then have $Y = y_0$, which is contrary to supposition, since $m > 1$.

We have, therefore, established the following theorem:

THEOREM VI. *If* $A_1, A_2, \ldots, A_m$ *are* m $(m > 1)$ *linearly independent linear homogeneous transformations belonging to an arbitrarily given domain of rationality R and constituting a hypercomplex number system with but one idempotent number, the modulus* 1, *there is then a system of just* $m-1$ *linearly independent linear functions,*

$$N_i = \tau_{i1} A_1 + \tau_{i2} A_2 + \ldots + \tau_{im} A_m, \qquad (i=1, 2 \ldots, m-1)$$

of the A's with coefficients rational with respect to R, constituting a maximal invariant nilpotent subsystem of $(A_1, A_2, \dots, A_m)$; and there is a number of the subsystem,

$$N = c_1 N_1 + c_2 N_2 + \dots + c_{m-1} N_{m-1},$$

where the c's are rational with respect to R, such that

$$N N_i = 0 = N_i N, \qquad (i = 1, 2, \dots, m-1).$$

Moreover, a linear substitution Ω belonging to R can be found such that

$$N_i = \Omega \begin{vmatrix} 0, & b_{12}^{(i)}, & \dots, & b_{1n}^{(i)} \\ 0, & 0, & \dots, & b_{2n}^{(i)} \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0 \end{vmatrix} \Omega^{-1},$$

where the b's are rational with respect to R. Whence it follows that the group $\overline{G}$ of transformations belonging to R constituted by the aggregate of transformations $\Sigma_{i=1}^{m} x_i A_i$ for all values of the x's rational with respect to R, is reducible, but not completely reducible, with respect to R; and is similar to the group of transformations belonging to R constituted by the aggregate of transformations defined by the equations

$$\xi'_\mu = y_0 \xi_\mu + \Sigma_{\nu=\mu+1}^{n} \Sigma_{i=1}^{m-1} y_i b_{\mu\nu}^{(i)} \xi_\nu, \qquad (\mu = 1, 2, \dots, n),$$

for all values of the parameters $y_0, y_1, \dots, y_{m-1}$ rational with respect to R.

A special case of this theorem is that for which $A_m = 1$, and $A_1, A_2, \dots, A_{m-1}$ constitute a nilpotent subsystem of $(A_1, A_2, \dots, A_m)$. For this system contains no idempotent number except the modulus 1, as may be shown as follows: Let

$$A = \alpha_m + \Sigma_{i=1}^{m-1} \alpha_i A_i = \Sigma_{i=1}^{m} \alpha_i A_i \neq 0$$

be idempotent. Then

$$A^2 = \alpha_m^2 + 2\alpha_m \Sigma_{i=1}^{m-1} \alpha_i A_i + (\Sigma_{i=1}^{m-1} \alpha_i A_i)^2 = \alpha_m + \Sigma_{i=1}^{m-1} \alpha_i A_i.$$

Therefore, $\alpha_m^2 = \alpha_m$, since $A_1, A_2, \dots, A_{m-1}$ constitute a subsystem of $(A_1, A_2, \dots, A_m)$. Since this subsystem is nilpotent, $\Sigma_{i=1}^{m-1} \alpha_i A_i$ is nilpotent. But there is no linear relation between the powers of a nilpotent number. Whence it follows that

$$(2\alpha_m - 1) \Sigma_{m=1}^{m-1} \alpha_i A_i = 0;$$

therefore, since $2\alpha_m - 1 \neq 0$,

$$\alpha_1 = \alpha_2 = \dots \alpha_{m-1} = 0,$$

and thus $A = \alpha_m A_m = \alpha_m 1$. Wherefore, $\alpha_m = 1$, and thus $A = A_m = 1$; otherwise, $A = 0$.